Math 2215 - Calculus 1

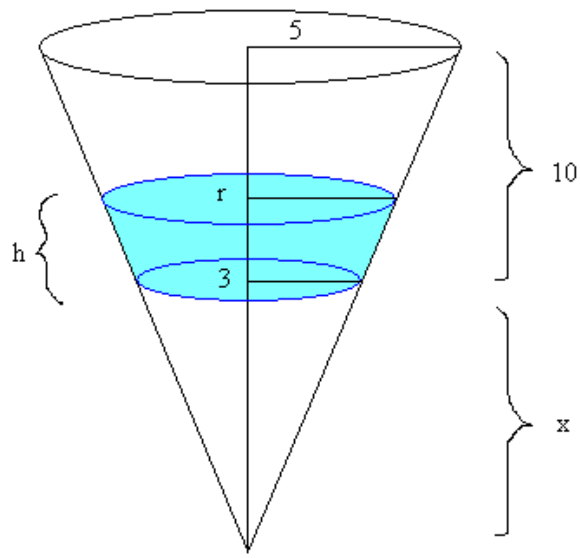
Homework #6 - 2005.10.14

Due Date - 2005.10.19

Solutions

A drinking glass is in the shape of a truncated cone with base of radius 3 cm, top radius 5 cm, and height 10 cm. A beverage is poured into the glass at a constant rate of 48 $\frac{\text{cm}^3}{\text{sec}}$. Find the rate at which the level of the beverage is rising in the glass when it is at a depth of 5 cm.

First we draw the following diagram. The reason why the rest of the cone is present will become clear soon.



If we define r to be the radius at the top of a volume V of beverage in the class with the depth h, then we need to find $\frac{dh}{dt}$ when $h = 5$ cm. We are given that $\frac{dV}{dt} = 48$ $\frac{\text{cm}^3}{\text{sec}}$. The first thing that must be done is to find a relationship between $\frac{dh}{dt}$ and $\frac{dV}{dt}$. This requires that we find a formula for the volume V of the blue shaded region in the above image.

Remember that the volume of a cone is given by the formula

$$Volume = \frac{1}{3}\pi\left(Radius\right)^2\left(Height\right).$$

To determine the volume of the shaded region, we must find the volume of two cones. The first is the cone with height $10 + x$ cm and radius 5 cm. The second is the cone with height x cm and radius 3 cm. We must now find x. This can be found by the ratio

$$\frac{5}{3} = \frac{10 + x}{x}.$$

This implies that $x = 15$ cm. We can now determine the volume of the beverage:

$$V = \frac{1}{3}\pi r^2 (15 + h) - \frac{1}{3}\pi \cdot 3^2 \cdot 15$$

Unfortunately, V is still a function of r and h. Using the fact that $x = 15$ and applying the method of similar triangles gives the relation

$$\frac{15}{3} = \frac{15 + h}{r}.$$

This yields that $r = 3 + \frac{h}{5}$. Substituting this into the volume formula gives

$$V = \frac{1}{3}\pi \left(3 + \frac{h}{5}\right)^2 (15 + h) - \frac{1}{3}\pi \cdot 3^2 \cdot 15 = \frac{5}{3}\pi \left(3 + \frac{h}{5}\right)^3 - \frac{1}{3}\pi \cdot 3^2 \cdot 15.$$

Taking a time derivative of both sides yields

$$\frac{dV}{dt} = 5\pi \left(3 + \frac{h}{5}\right)^2 \cdot \frac{1}{5}\frac{dh}{dt}.$$

Substituting in $\frac{dV}{dt} = 48$ and $h = 5$ into the above, and solving for $\frac{dh}{dt}$ gives

$$\frac{dh}{dt} = \frac{3}{\pi}\frac{\text{cm}}{\text{sec}}$$