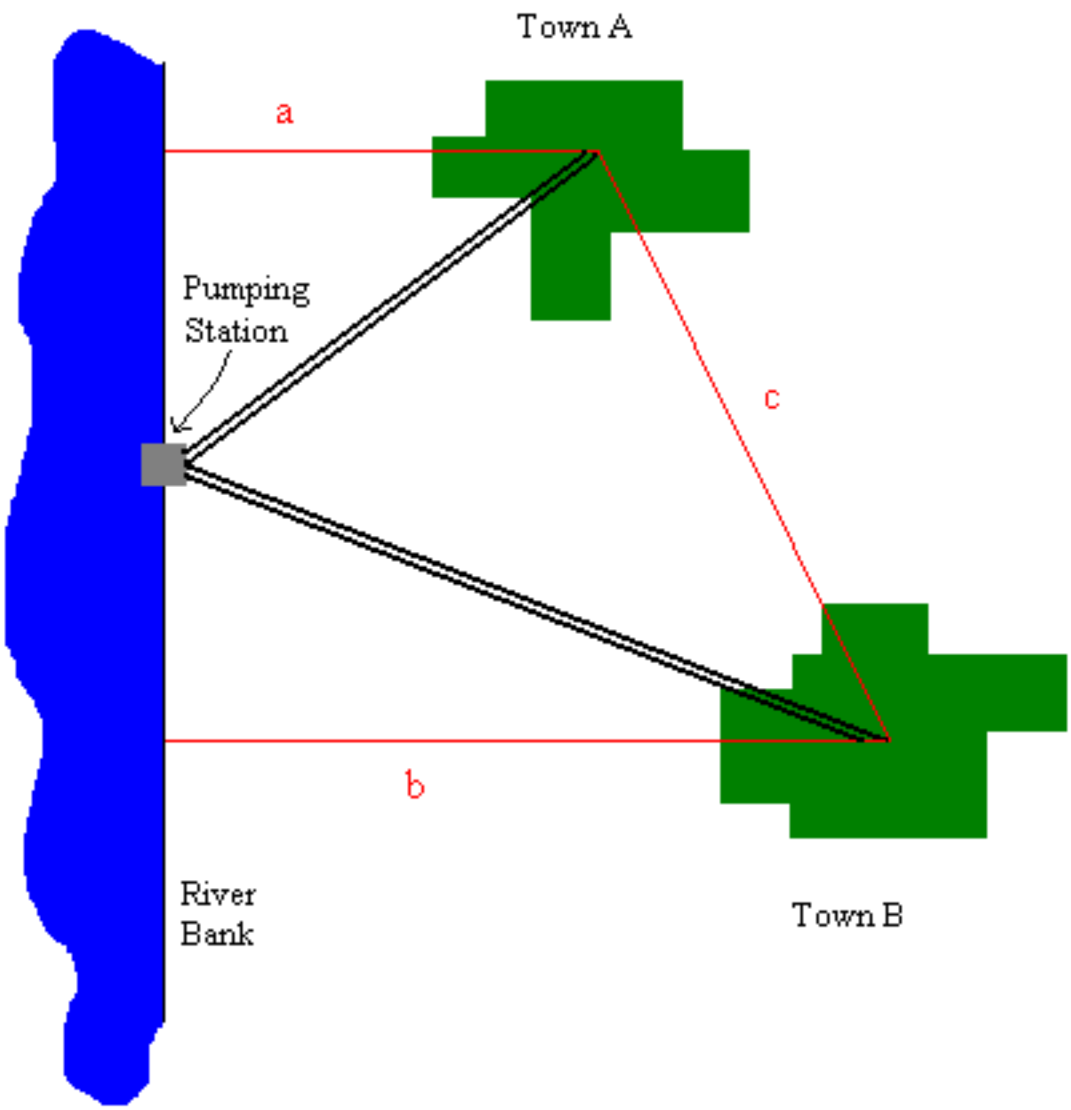
Math 2215 - Calculus 1

Homework #7 - 2005.11.16

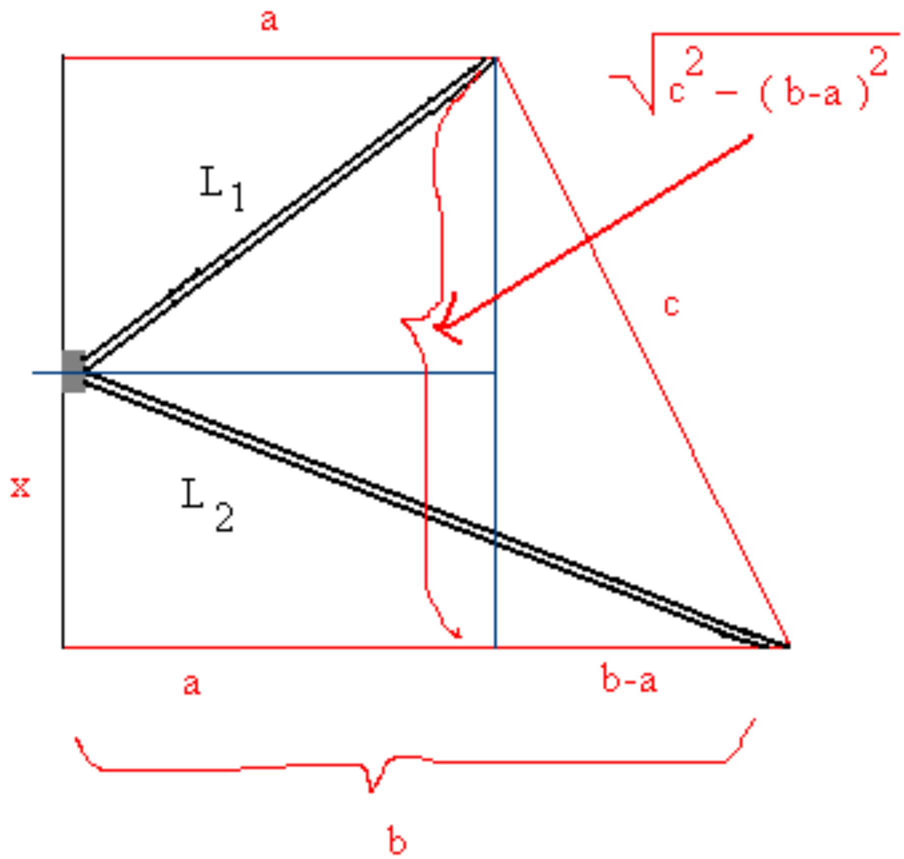
Due Date - 2005.11.28

Solutions

Two towns, located on the same side of a straight river agree to construct a pumping station and filtering plant at the river's edge, t be used jointly to supply the towns with water. The distances of the two towns from the river are a and b and the distance between them is c. Show that the sum of the lengths of the pipe lines joining them to the pumping station is at least as great as $\sqrt{c^2 + 4ab}$.



Consider the following diagram:



Then we want to show that the minimum the distance $L_1 + L_2$ can be is $\sqrt{c^2 + 4ab}$.

Using the above diagram, and noting the x measures the distance along the river from the pumping station to the point which is due west of town B, we have that

$$L(x) = L_1(x) + L_2(x) = \sqrt{a^2 + c^2 - (b-a)^2 - 2\sqrt{c^2 - (b-a)^2}x + x^2} + \sqrt{b^2 + x^2}.$$

Here we notice that the domain of $L(x)$ is $x \in [0, \sqrt{c^2 - (b-a)^2}]$. So we will be looking for critical points in the domain of $L(x)$.

Taking a derivative gives:

$$L'(x) = \frac{x - \sqrt{c^2 - (b-a)^2}}{L_1} + \frac{x}{L_2},$$

which under a common denominator is

$$L'(x) = \frac{\sqrt{b^2+x^2}\left(x - \sqrt{c^2-(b-a)^2}\right) + x\sqrt{a^2+c^2-(b-a)^2-2\sqrt{c^2-(b-a)^2}x+x^2}}{L_1 L_2}.$$

Setting $L' = 0$, and noting that L_1 and L_2 are always positive yields the equation

$$\sqrt{b^2+x^2}\left(x - \sqrt{c^2-(b-a)^2}\right) + x\sqrt{a^2+c^2-(b-a)^2-2\sqrt{c^2-(b-a)^2}x+x^2} = 0,$$

which we rewrite as

$$-\sqrt{b^2+x^2}\left(x - \sqrt{c^2-(b-a)^2}\right) = x\sqrt{a^2+c^2-(b-a)^2-2\sqrt{c^2-(b-a)^2}x+x^2}.$$

This is where our extreme mastery of algebra comes in handy. Setting $r = c^2 - (b-a)^2$, notice that the above equation can be expressed as

$$-\sqrt{b^2+x^2}\left(x-\sqrt{r}\right) = x\sqrt{a^2+\left(x-\sqrt{r}\right)^2}.$$

Squaring both sides gives:

$$\left(b^2+x^2\right)\left(x-\sqrt{r}\right)^2 = x^2\left(a^2+\left(x-\sqrt{r}\right)^2\right).$$

Notice that some terms cancel. This simplifies things to:

$$b^2\left(x-\sqrt{r}\right)^2 = a^2x^2.$$

Expanding and combining like terms yields:

$$\left(b^2-a^2\right)x^2 - 2b^2\sqrt{r}x + b^2r = 0.$$

Applying the quadratic formula gives:

$$x = \frac{2b^2\sqrt{r} \pm \sqrt{4b^4r - 4\left(b^2-a^2\right)b^2r}}{2\left(b^2-a^2\right)}.$$

Some serious simplifications can be done to get:

$$x_\pm = \frac{b\sqrt{r}\left(b \pm a\right)}{b^2-a^2}.$$

When using the $(+)$ sign, notice that the above simplifies to

$$x_+ = \frac{b\sqrt{r}}{b-a},$$

and similarly when using the $-$ sign one gets

$$x_- = \frac{b\sqrt{r}}{b+a}.$$

The next question to ask is, which one really works? By squaring the equation to get things to work, perhaps one has introduced extra solution. This can be seen to be true as follows. Remember that before we squared everything, we were looking for solutions to:

$$-\sqrt{b^2+x^2}\left(x-\sqrt{r}\right) = x\sqrt{a^2+\left(x-\sqrt{r}\right)^2}.$$

Substituting in $x = x_+ = \frac{b\sqrt{r}}{b-a}$ gives:

$$-\sqrt{b^2+\frac{b^2 r}{(b-a)^2}}\left(\frac{b\sqrt{r}}{b-a}-\sqrt{r}\right) = \frac{b\sqrt{r}}{b-a}\sqrt{a^2+\left(\frac{b\sqrt{r}}{b-a}-\sqrt{r}\right)^2}.$$

After a few steps of algebraic manipulation, one has:

$$-ab\sqrt{r}\sqrt{1+\frac{r}{(b-a)^2}} = ab\sqrt{r}\sqrt{1+\frac{r}{(b-a)^2}},$$

clearly a contradiction. However, notice that if we use $x = x_- = \frac{b\sqrt{r}}{b+a}$, we do indeed get the correct relation:

$$ab\sqrt{r}\sqrt{1+\frac{r}{(b+a)^2}} = ab\sqrt{r}\sqrt{1+\frac{r}{(b+a)^2}}.$$

Furthermore, it is easily seen that x_- is in the domain of $L(x)$ since

$$0 < \frac{b\sqrt{r}}{b+a} < \sqrt{r}.$$

Next notice that when substituting in the above value for $x = x_-$ one has:

$$L\left(\frac{b\sqrt{r}}{b+a}\right) = \sqrt{a^2+r-2\frac{rb}{b+a}+\frac{b^2 r}{(b+a)^2}} + \sqrt{b^2+\frac{b^2 r}{(b+a)^2}}.$$

Getting common denominator of $(b+a)^2$ in both terms (and thus pulling out the $b+a$ term in both denominators after taking square roots, gives

$$L(x_-) = \frac{\sqrt{(b+a)^2(a^2+r)-2rb(b+a)+b^2 r}}{b+a} + \frac{\sqrt{b^2(b+a)^2+rb^2}}{b+a}.$$

Combining similar terms in each piece and simplifying gives:

$$= \frac{a}{b+a}\sqrt{(a+b)^2+r} + \frac{b}{b+a}\sqrt{(a+b)^2+r}$$

which yields (after expanding the r term again):

$$\sqrt{(a+b)^2+r} = \sqrt{c^2+4ab}$$

To show that this is a minimum, one could use a geometric argument to show that the value at x_- must be the minimum. However, this can be done algebraically as well. When $x = 0$, notice that this corresponds to the pumping station being located due west of town B.

$$L(0) = L_1(0) + L_2(0) = \sqrt{a^2+c^2-(b-a)^2} + b = \sqrt{c^2-b^2+2ab} + b.$$

We need to show that $L(0) > \sqrt{c^2 + 4ab}$. So, we must show:

$$\sqrt{c^2 - b^2 + 2ab} + b > \sqrt{c^2 + 4ab}.$$

Squaring both sides and simplifying gives:

$$2ab + 2b\sqrt{c^2 - b^2 + 2ab} > 4ab,$$

which simplifies to

$$\sqrt{c^2 - b^2 + 2ab} > a.$$

Squaring both sides and factoring gives

$$c^2 - (b - a)^2 > 0.$$

This of course we know to be true, as it is the square of the vertical distance between the two towns (see the second figure). Since we arrived at a true statement, the original statement must also be true. Therefore, $L(0) > \sqrt{c^2 + 4ab}$. A similar argument can be used when the pumping station is ,located due west of town A as well. Thus the minimum does occur at the given critical point, and the minimum value is as stated in the problem.