Math 3113 - Multivariable Calculus

Final Exam - 2006.04.26

Due Date - 2006.05.08 by 11:00 AM

Name: ______________________

Please answer each question as fully as possible. Partial credit will only be given if you show your work. Also, make sure your work is neat and orderly. Finally, DO NOT work together, DO NOT get assistance from anyone but me!

1. Let $\overrightarrow{v_1} = \langle 1, 2, 3 \rangle$ and $\overrightarrow{v_2} = \langle 2, 1, -1 \rangle$. Answer the following questions.

a) Find a unit vector $\overrightarrow{w}$ which is orthogonal to both $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

b) Find a vector $\overrightarrow{r}$ which lies in the plane spanned by $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

c) Since there were many answers to part b, can you find a $\overrightarrow{r}$ such that $\overrightarrow{r}$ points in the same direction as $\overrightarrow{w}$?

d) Compute $proj_{\overrightarrow{v_1}}\overrightarrow{v_2}$ and $proj_{\overrightarrow{v_2}}\overrightarrow{v_1}$.

e) What is the value of $comp_{\overrightarrow{v_1}}\overrightarrow{w}$ and why?

2. Convert the point $(1, -1, 1)$ in rectangular coordinates into both cylindrical and spherical coordinates.

3. Show that the two planes $ax+by+cz+d_1 = 0$ and $ax+by+cz+d_2 = 0$ are parallel.

4. It is known that the distance D between parallel places (given as in problem 3) is given by the formula

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Find the equations of the two planes that are parallel to the plane $4x-3y+5z-\sqrt{2} = 0$ and are 3 units away from it.

5. Consider the parametric equations for the ellipse, $x = a\cos(t)$, $y = b\sin(t)$ with $a > b > 0$.

a) Compute the curvature κ of the ellipse as defined above.

b) Show that the curvature κ of the ellipse is never zero.

c) Prove that the points of maximum curvature of an ellipse occur at the intersection with the $x-$axis and that the value of the maximum curvature is $\kappa_{max} = a/b^2$.

d) Prove that the points of maximum curvature of an ellipse occur at the intersection with the $y-$axis and that the value of the maximum curvature is $\kappa_{min} = b/a^2$.

6. Let $w = f(x, y)$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

a) Show that

$$\frac{\partial w}{\partial r} = f_x \cos(\theta) + f_y \sin(\theta)$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin(\theta) + f_y \cos(\theta).$$

b) Solve the equations in part a to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \theta$.

c) Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

7. Let $f(x, y) = x^3 y + xy^2 - x + y$ with $x = r^2 \sin(\theta)$ and $y = r^3 \cos^2(\theta)$.

a) Compute $\frac{\partial f}{\partial r}$.

b) Compute $\frac{\partial f}{\partial \theta}$.

c) Compute $\frac{\partial^2 f}{\partial \theta \partial r}$.

d) Compute $\frac{\partial^2 f}{\partial r^2}$.

8. Find an equation for the plane which is tangent to the surface

$$\cos(\pi x) - x^2 y + e^{xy} + yz = 4$$

at the point $P = (0, 1, 2)$.

9. Consider the function $f(x, y) = x^2 - 3xy + 4y^2$ at the point $P = (1, 0)$.

a) Is there a direction $\overrightarrow{a}$ in which the rate of change of $f(x, y)$ at P is zero? If so, what is $\overrightarrow{a}$? If not, why?

b) Is there a direction $\overrightarrow{a}$ in which the rate of change of $f(x, y)$ at P is 16? If so, what is $\overrightarrow{a}$? If not, why?

10. The following integral is over a region in the Cartesian coordinate plane. Sketch the region and then evaluate the integral.

$$\int_{-\pi/3}^{\pi/3} \int_0^{\sec(x)} 3\cos(x) dy dx$$

11. Evaluate

$$\int \int_R e^{x^2+y^2} dy dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$.

12. The average value A of a function F over a region D in space is defined by the formula

$$A = \frac{1}{V(D)} \int \int_D \int F \; dV,$$

where $V(D)$ is the volume of the region D. Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and $x = 2$, $y = 2$, and $z = 2$.

13. How much of the assigned textbook homework did you actually do over the course of the semester? Please answer honestly, your grade will not be affected either way.