Math 3113 - Multivariable Calculus

Homework #1 - 2006.01.18

Due Date - 2006.01.25

Solutions

1. Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, show that the point M defined as below is equidistant from P_1 and P_2.

$$M\left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2)\right).$$

The distance between P_1 and M is defined to be

$$d(P_1, M) = \sqrt{\frac{1}{4}(x_1-x_2)^2 + \frac{1}{4}(y_1-y_2)^2 + \frac{1}{4}(z_1-z_2)^2}.$$

The distance between P_2 and M is defined to be

$$d(P_2, M) = \sqrt{\frac{1}{4}(x_2-x_1)^2 + \frac{1}{4}(y_2-y_1)^2 + \frac{1}{4}(z_2-z_1)^2}.$$

Since $(a-b)^2 = (b-a)^2$, one can conclude that $d(P_1, M) = d(P_2, M)$. Furthermore, notice that

$$\sqrt{\frac{1}{4}(x_2-x_1)^2 + \frac{1}{4}(y_2-y_1)^2 + \frac{1}{4}(z_2-z_1)^2} = \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2},$$

which is exactly half the distance between P_1 and P_2.

2. Using the definitions of P_1, P_2 and M as in the previous problem, compute $\overrightarrow{P_1M}$ and $\overrightarrow{P_1P_2}$.

$$\overrightarrow{P_1P_2} = (x_2-x_1, y_2-y_1, z_2-z_1)$$

and

$$\overrightarrow{P_1M} = \left(\frac{1}{2}(x_1+x_2)-x_1, \frac{1}{2}(y_1+y_2)-y_1, \frac{1}{2}(z_1+z_2)-z_1\right) = \frac{1}{2}(x_2-x_1, y_2-y_1, z_2-z_1).$$

Notice that $\overrightarrow{P_1M} = \frac{1}{2}\overrightarrow{P_1P_2}$.

3. Remember that $\overrightarrow{P_1M} \cdot \overrightarrow{P_1P_2} = \left|\overrightarrow{P_1M}\right| \left|\overrightarrow{P_1P_2}\right| \cos(\theta)$. What is θ in this case?

$$\cos(\theta) = \frac{\overrightarrow{P_1M} \cdot \overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1M}\right| \left|\overrightarrow{P_1P_2}\right|} = \frac{\frac{1}{2}\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_2}}{\left|\frac{1}{2}\overrightarrow{P_1P_2}\right| \left|\overrightarrow{P_1P_2}\right|} = \frac{\frac{1}{2}\overrightarrow{P_1P_2}}{\left|\frac{1}{2}\overrightarrow{P_1P_2}\right|} \cdot \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{\frac{1}{2}}{\frac{1}{2}} \frac{\left|\overrightarrow{P_1P_2}\right|^2}{\left|\overrightarrow{P_1P_2}\right|^2} = 1.$$

Thus $\theta = 0$.

4. Using the above three problems, show that the point M is the midpoint on the line segment connecting the points P_1 and P_2.

Clearly since problem 3 shows that $\theta = 0$, the two vectors $\overrightarrow{P_1M}$ and $\overrightarrow{P_1P_2}$ are parallel.

Secondly, since the two vectors mentioned above share a same end point, one must conclude that they do indeed lie along the same line.

Furthermore, from problem 1, it is clear that the distance between the points P_1 and M is the same as that of P_2 and M. Furthermore, notice that all three coordinates of the point M lie in between those of P_1 and P_2 (which is really required by the previous two statements anyways).

Thus we must conclude that the point M is the midpoint on the line segment connecting the points P_1 and P_2.

5. Suppose that M, P_1 and P_2 are defined as previous. Let $A(a_1, a_2, a_3)$ be any other point in $\mathbb{R}^3$. Show that $\overrightarrow{AM} = \frac{1}{2}\left(\overrightarrow{AP_1} + \overrightarrow{AP_2}\right)$.

This is obviously true since $M = \frac{1}{2}(P_1 + P_2)$. I.e.

$$\frac{1}{2}\left(\overrightarrow{AP_1} + \overrightarrow{AP_2}\right) = \frac{1}{2}\overrightarrow{A(P_1+P_2)} = \overrightarrow{AM}.$$