Math 3113 - Multivariable Calculus

Homework #2 - 2006.01.26
Due Date - 2006.02.01
Solutions

Let $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$, $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$ and $\overrightarrow{c} = \langle c_1, c_2, c_3 \rangle$ be vectors. Prove the following:

1. $\left(\overrightarrow{a} - \overrightarrow{b}\right) \times \left(\overrightarrow{a} + \overrightarrow{b}\right) = 2\left(\overrightarrow{a} \times \overrightarrow{b}\right)$

$$\begin{aligned}\left(\overrightarrow{a} - \overrightarrow{b}\right) \times \left(\overrightarrow{a} + \overrightarrow{b}\right) &= (\overrightarrow{a} - \overrightarrow{b}) \times a + (\overrightarrow{a} - \overrightarrow{b}) \times \overrightarrow{b} \\ &= \overrightarrow{a} \times \overrightarrow{a} - \overrightarrow{b} \times \overrightarrow{a} + \overrightarrow{a} \times \overrightarrow{b} - \overrightarrow{b} \times \overrightarrow{b}\end{aligned}$$

Using the facts that $\overrightarrow{v} \times \overrightarrow{v} = 0$ and $-\overrightarrow{b} \times \overrightarrow{a} = \overrightarrow{a} \times \overrightarrow{b}$, one arrives at the solution.

2. $\left|\overrightarrow{a} \times \overrightarrow{b}\right|^2 + \left|\overrightarrow{a} \cdot \overrightarrow{b}\right|^2 = |\overrightarrow{a}|^2 \left|\overrightarrow{b}\right|^2$

$$\begin{aligned}\left|\overrightarrow{a} \times \overrightarrow{b}\right|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - 2a_1b_1a_2b_2 - 2a_1b_1a_3b_3 - 2a_2b_2a_3b_3 \\ |\overrightarrow{a}|^2 \left|\overrightarrow{b}\right|^2 &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) \\ &= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 \\ \left|\overrightarrow{a} \cdot \overrightarrow{b}\right|^2 &= (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= a_1^2b_1^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + a_2^2b_2^2 + 2a_2b_2a_3b_3 + a_3^2b_3^2\end{aligned}$$

Clearly, from the above expansions, the equation holds.

3. $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} - (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{c}$

$$\begin{aligned}\overrightarrow{b} \times \overrightarrow{c} &= \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle \\ \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) &= \langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \rangle \\ &= \langle a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \rangle\end{aligned}$$

The above is the right hand side of the equation. Next we compute the left hand side.

$$\begin{aligned}(\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} &= \langle (a_1c_1 + a_2c_2 + a_3c_3)b_1, (a_1c_1 + a_2c_2 + a_3c_3)b_2, (a_1c_1 + a_2c_2 + a_3c_3)b_3 \rangle \\ &= \langle a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3, a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3, a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 \rangle \\ (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{c} &= \langle (a_1b_1 + a_2b_2 + a_3b_3)c_1, (a_1b_1 + a_2b_2 + a_3b_3)c_2, (a_1b_1 + a_2b_2 + a_3b_3)c_3 \rangle \\ &= \langle a_1b_1c_1 + a_2b_2c_1 + a_3b_3c_1, a_1b_1c_2 + a_2b_2c_2 + a_3b_3c_2, a_1b_1c_3 + a_2b_2c_3 + a_3b_3c_3 \rangle\end{aligned}$$

Taking the difference of the above two equations, one has the left hand side given by:

$$\begin{aligned}&(\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} - (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{c} = \\ &\langle a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1, a_1b_2c_1 - a_1b_1c_2 + a_3b_2c_3 - a_3b_3c_2, a_1b_3c_1 - a_1b_1c_3 + a_2b_3c_2 - a_2b_2c_3 \rangle\end{aligned}$$

From the expansion, we see that $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is identical to $(\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} - (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{c}$

4. $\overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c}\right) + \overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) + \overrightarrow{c} \times \left(\overrightarrow{a} \times \overrightarrow{b}\right) = 0$

From problem 3, we have the following equalities.

$$\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{c} \cdot \overrightarrow{a})\overrightarrow{b} - (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{c}$$

$$\overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) = (\overrightarrow{a} \cdot \overrightarrow{b})\overrightarrow{c} - (\overrightarrow{c} \cdot \overrightarrow{b})\overrightarrow{a}$$

$$\overrightarrow{c} \times (\overrightarrow{a} \times \overrightarrow{b}) = (\overrightarrow{b} \cdot \overrightarrow{c})\overrightarrow{a} - (\overrightarrow{a} \cdot \overrightarrow{c})\overrightarrow{b}$$

Substituting these into the problem gives the desired result.