Math 3113 - Multivariable Calculus

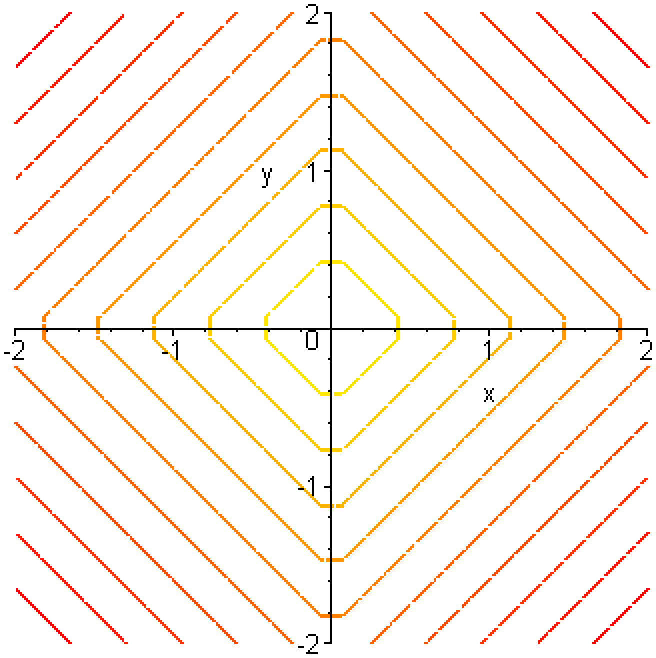
Homework #6 - 2006.03.23

Due Date - 2006.03.29

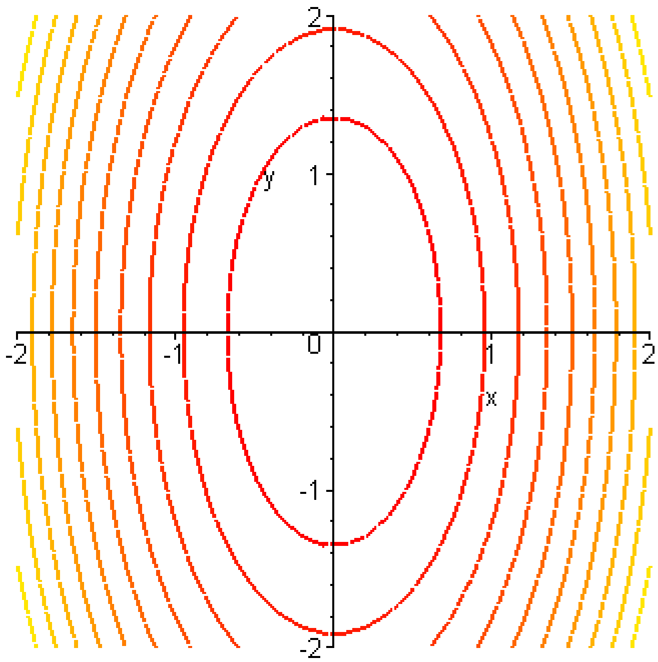
Solutions

1. Sketch a contour map of each of the following functions.

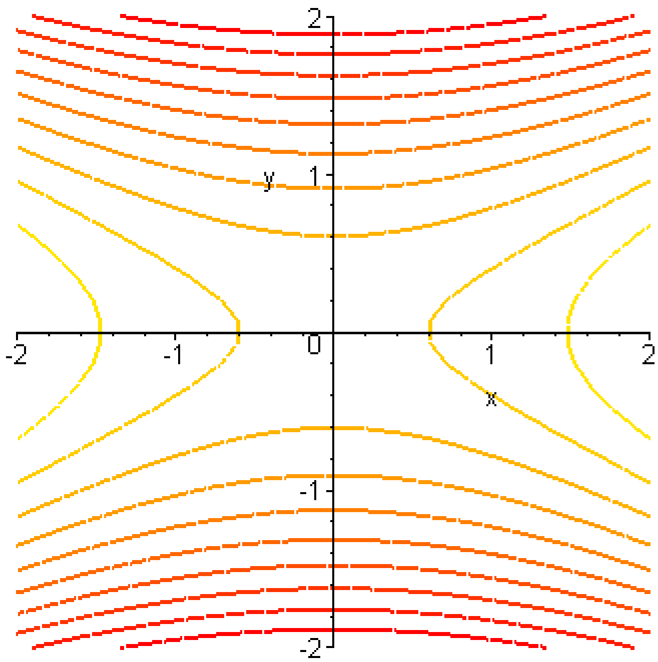
a) $f(x, y) = 1 - |x| - |y|$



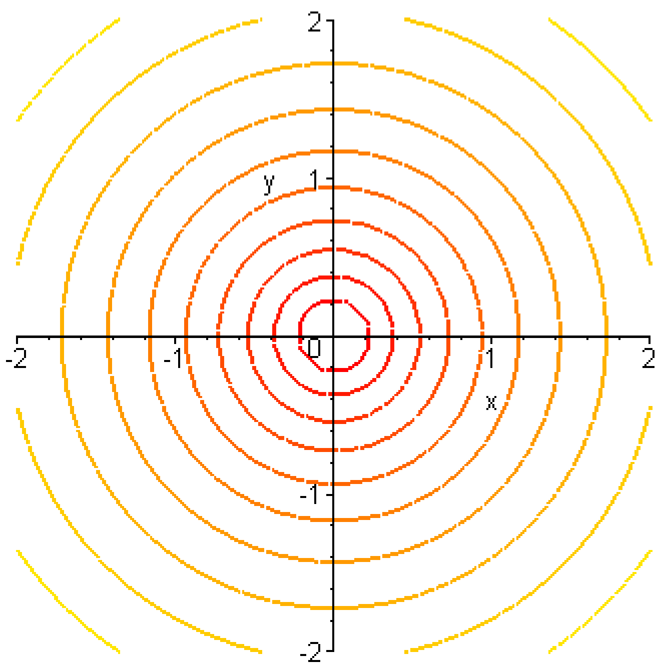
b) $f(x, y) = 4x^2 + y^2$



c) $f(x,y) = x^2 - 4y^2$



d) $f(x,y) = \ln\left(\sqrt{x^2+y^2}+1\right)$



2. Consider the function

$$f(x,y) = \frac{x^2(x+y)}{x^4+(x+y)^2}.$$

a) Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ for all curves of the form $y = ax^k$ for $k \geq 1$.

By setting $y = ax^k$, we have the following limit:

$$\lim_{x\to 0} \frac{x^2(x+ax^k)}{x^4+(x+ax^k)^2}.$$

Factoring an x^3 out of the numerator, and an x^2 out of the denominator gives

$$\lim_{x\to 0} \frac{x^2(x+ax^k)}{x^4+(x+ax^k)^2} = \lim_{x\to 0} \frac{x^3\left(1+ax^{k-1}\right)}{x^2\left(x^2+(1+ax^{k-1})^2\right)} = \lim_{x\to 0} \frac{x\left(1+ax^{k-1}\right)}{x^2+(1+ax^{k-1})^2}$$

Since $k \geq 1$, one has that $\lim_{x\to 0} x^{k-1} = 0$, and therefore

$$\lim_{x\to 0} \frac{x\left(1+ax^{k-1}\right)}{x^2+(1+ax^{k-1})^2} = \frac{0}{1} = 0.$$

b) Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ for all curves of the form $x = ay^k$ for $k \geq 1$.

By setting $x = ay^k$, we have the following limit:

$$\lim_{y\to 0} \frac{\left(ay^k\right)^2\left(ay^k+y\right)}{\left(ay^k\right)^4+\left(ay^k+y\right)^2}.$$

Factoring a y^3 out of the numerator, and a y^2 out of the denominator gives

$$\lim_{y\to 0} \frac{\left(ay^k\right)^2\left(ay^k+y\right)}{\left(ay^k\right)^4+\left(ay^k+y\right)^2} = \lim_{y\to 0} \frac{y^3\left(a^3y^{3(k-1)}+a^2y^{2(k-1)}\right)}{y^2\left(a^4y^{4k-2}+a^2y^{2(k-1)}+ay^{k-1}+1\right)} = \lim_{y\to 0} \frac{y\left(a^3y^{3(k-1)}+a^2y^{2(k-1)}\right)}{a^4y^{4k-2}+a^2y^{2(k-1)}+ay^{k-1}+1}.$$

Using similar arguments to part a), one arrives at the same answer of

$$\lim_{y\to 0} \frac{y\left(a^3y^{3(k-1)}+a^2y^{2(k-1)}\right)}{a^4y^{4k-2}+a^2y^{2(k-1)}+ay^{k-1}+1} = \frac{0}{1} = 0.$$

c) Show that $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $y = x^2 - x$.

Here we have

$$\lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

3. Notice that the following two expressions are equivalent:

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x+a, y+b)$$

Since it is easier to work with curves through the origin, in limits where $(a,b) \neq (0,0)$ and the limit is not obvious, one can replace x by $x+a$ and y by $y+b$ and let $(x,y) \to (0,0)$.

Consider the following limit:

$$\lim_{(x,y)\to(1,0)} \frac{4(x-1)^4(y+1)+4y^2}{(x-1)^4+y^2}.$$

a) Rewrite the above limit so that the limit goes through the origin instead of through the point $(1,0)$.

$$\lim_{(x,y)\to(1,0)} \frac{4(x-1)^4(y+1)+4y^2}{(x-1)^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^4(y+1)+4y^2}{x^4+y^2}.$$

b) Using the STRICT definition of the limit, show that the limit in part a) is equal to 4.

We require that

$$\left|\frac{4x^4(y+1)+4y^2}{x^4+y^2} - 4\right| \leq \varepsilon$$

whenever

$$\sqrt{x^2+y^2} \leq \delta.$$

Notice that

$$\left|\frac{4x^4(y+1)+4y^2}{x^4+y^2}-4\right| = \left|4\frac{x^4y}{x^4+y^2}\right|.$$

Since $y^2 \geq 0$, one has that

$$\left|\frac{x^4y}{x^4+y^2}\right| \leq |y|\left|\frac{x^4+y^2}{x^4+y^2}\right| = |y|.$$

Thus,

$$\left|4\frac{x^4y}{x^4+y^2}\right| \leq 4|y| = 4\sqrt{y^2} \leq 4\sqrt{x^2+y^2}.$$

So if $\delta = \frac{\varepsilon}{4}$, one has that

$$\left|\frac{4x^4(y+1)+4y^2}{x^4+y^2}-4\right| \leq \varepsilon$$

whenever

$$\sqrt{x^2+y^2} \leq \delta = \frac{\varepsilon}{4}.$$

c) What value does δ have to be smaller than if $\varepsilon = \frac{1}{25}$?
From part b), $\delta \leq \frac{1}{100}$.