Math 3113 - Multivariable Calculus

Homework #7 - 2006.03.29

Due Date - 2006.04.03

Solutions

1. Consider the three dimensional Laplace equation given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Which of the following functions satisfy the Laplace equation?
a) $f(x,y,z) = x^2 + y^2 - 2z^2$
b) $f(x,y,z) = 2z^3 - 3\left(x^2 + y^2\right) z$
c) $f(x,y,z) = e^{-2y}\cos(2x) + z$
d) $f(x,y,z) = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$
e) $f(x,y,z) = e^{3x+4y}\cos(5z)$.

All of the functions are solutions.

2. Consider the one dimensional wave equation given by

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is time and c is the velocity of the wave being propagated. Which of the following functions satisfy the wave equation?
a) $w(x,t) = \sin(x + ct)$
b) $w(x,t) = 5\cos(3x + 3ct) + e^{x+ct}$
c) $w(x,t) = \sin(x + ct) + \cos(2x + 2ct)$
d) $w(x,t) = \ln(2x + 2ct)$
e) $w(x,t) = \tan(2x - 2ct)$

All of the functions are solutions.

3. Show that the if $f(z)$ is a differentiable function, then $u = f(z(x,t))$ where $z = x - t$, will always satisfy the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0,$$

independent of the actual form of f.

Here, first we compute $\frac{\partial u}{\partial x}$. Notice that

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = \frac{df}{dz}\frac{dz}{dx} = f'(z) \cdot 1$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t} = \frac{df}{dz}\frac{dz}{dt} = f'(z) \cdot (-1).$$

Putting these two together gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = f'(z) - f'(z) = 0.$$

4. Show that the if $f(z)$ is a differentiable function, then $w = f(z(x,t))$ where $z = x + ct$, will always satisfy the one dimensional wave equation given in problem 2.
We will start similarly to problem 3.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = \frac{df}{dz}\frac{dz}{dx} = f'(z) \cdot 1,$$

and thus

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{d}{dx} f'(z(x,t) = \frac{df'}{dz}\frac{dz}{dx} = f''(z) \cdot 1 = f''(x + ct).$$

Also,

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial t} = \frac{df}{dz}\frac{dz}{dt} = f'(z) \cdot c,$$

and

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} = c\frac{d}{dt}f'(z(x,t) = c\frac{df'}{dz}\frac{dz}{dt} = cf''(z) \cdot c = c^2 f''(x+ct).$$

The wave equation states

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

and in our case

$$\frac{\partial^2 w}{\partial t^2} = c^2 f''(z) = c^2 \frac{\partial^2 w}{\partial x^2}$$

which is exactly as desired.