# Math 3113 - Multivariable Calculus

## Homework #8 - 2006.04.05
## Due Date - 2006.04.12
## Solutions

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A function $f(x, y)$ is said to be homogeneous of degree $n$ if $f(tx, ty) = t^n f(x, y)$ for all $t$. Here $n$ is a positive integer. Homogeneous functions are very important in the study of elliptic curves and cryptography.

1. Show that the function $r(x, y) = 4xy^6 - 2x^3y^4 + x^7$ is homogeneous of degree 7.

$$r(tx, ty) = 4txt^6y^6 - 2t^3x^3t^4y^4 + t^7x^7 = 4t^7xy^6 - 2t^7x^3y^4 + t^7x^7 = t^7r(x, y).$$

2. Give a nontrivial example of a function $g(x, y)$ which is homogeneous of degree 9.

Answers will vary.

3. Show that if $f(x, y)$ is homogeneous of degree $n$ and sufficiently differentiable, then $f(x, y)$ satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

(Hint: let $X = tx$ and $Y = ty$ and take the derivative with respect to $t$ of the equation $f(tx, ty) = t^n f(x, y)$ and consider the case of $t = 1$.)

Using the hint, we have

$$\frac{\partial}{\partial t}f(X, Y) = \frac{\partial}{\partial t}t^n f(x, y),$$

where

$$\frac{\partial}{\partial t}f(X, Y) = \frac{\partial f(X, Y)}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial f(X, Y)}{\partial Y}\frac{\partial Y}{\partial t} = \frac{\partial f(X, Y)}{\partial X}x + \frac{\partial f(X, Y)}{\partial Y}y$$

and

$$\frac{\partial}{\partial t}t^n f(x, y) = nt^{n-1}f(x, y).$$

Setting these two equal gives

$$\frac{\partial f(X, Y)}{\partial X}x + \frac{\partial f(X, Y)}{\partial Y}y = nt^{n-1}f(x, y).$$

Notice that if $t =$, then $X = x$ and $Y = y$, therefore

$$\frac{\partial f(x, y)}{\partial x}x + \frac{\partial f(x, y)}{\partial y}y = nt^{n-1}f(x, y).$$

4. Using the assumptions in problem 3, show that $f(x, y)$ also satisfies

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial x^2} = n(n-1)f(x, y).$$

Here we simply take another time derivative:

$$\frac{\partial}{\partial t}\left(\frac{\partial f(X, Y)}{\partial X}x + \frac{\partial f(X, Y)}{\partial Y}y\right) = \frac{\partial}{\partial t}nt^{n-1}f(x, y),$$

where

$$\begin{aligned}\frac{\partial}{\partial t}\left(x\frac{\partial f(X, Y)}{\partial X}\right) =& x\left[\frac{\partial^2 f(X, Y)}{\partial X^2}\frac{\partial X}{\partial t} + \frac{\partial^2 f(X, Y)}{\partial X \partial Y}\frac{\partial Y}{\partial t}\right] \\ =& x\left[\frac{\partial^2 f(X, Y)}{\partial X^2}x + \frac{\partial^2 f(X, Y)}{\partial X \partial Y}y\right] \\ =& x^2\frac{\partial^2 f(X, Y)}{\partial X^2} + xy\frac{\partial^2 f(X, Y)}{\partial X \partial Y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t}\left(y\frac{\partial f(X,Y)}{\partial Y}\right) =&y\left[\frac{\partial^2 f(X,Y)}{\partial Y^2}\frac{\partial Y}{\partial t}+\frac{\partial^2 f(X,Y)}{\partial Y\partial X}\frac{\partial X}{\partial t}\right]\\ =&y\left[\frac{\partial^2 f(X,Y)}{\partial Y^2}y+\frac{\partial^2 f(X,Y)}{\partial Y\partial X}x\right]\\ =&y^2\frac{\partial^2 f(X,Y)}{\partial Y^2}+xy\frac{\partial^2 f(X,Y)}{\partial Y\partial X}.\end{aligned}$$

But notice that

$$\frac{\partial^2 f(X,Y)}{\partial X\partial Y}=\frac{\partial^2 f(X,Y)}{\partial Y\partial X},$$

therefore,

$$\frac{\partial}{\partial t}\left(\frac{\partial f(X,Y)}{\partial X}x+\frac{\partial f(X,Y)}{\partial Y}y\right)=x^2\frac{\partial^2 f(X,Y)}{\partial X^2}+2xy\frac{\partial^2 f(X,Y)}{\partial X\partial Y}+y^2\frac{\partial^2 f(X,Y)}{\partial Y^2}.$$

Next, the left hand side:

$$\frac{\partial}{\partial t}nt^{n-1}f(x,y)=n(n-1)t^{n-2}f(x,y).$$

Setting these equal we have

$$x^2\frac{\partial^2 f(X,Y)}{\partial X^2}+2xy\frac{\partial^2 f(X,Y)}{\partial X\partial Y}+y^2\frac{\partial^2 f(X,Y)}{\partial Y^2}=n(n-1)t^{n-2}f(x,y).$$

Once again, when $t=1$, one arrives at

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f(x,y).$$