Math 2143 - Brief Calculus with Applications

Exam #1 - 2014.09.23

Solutions

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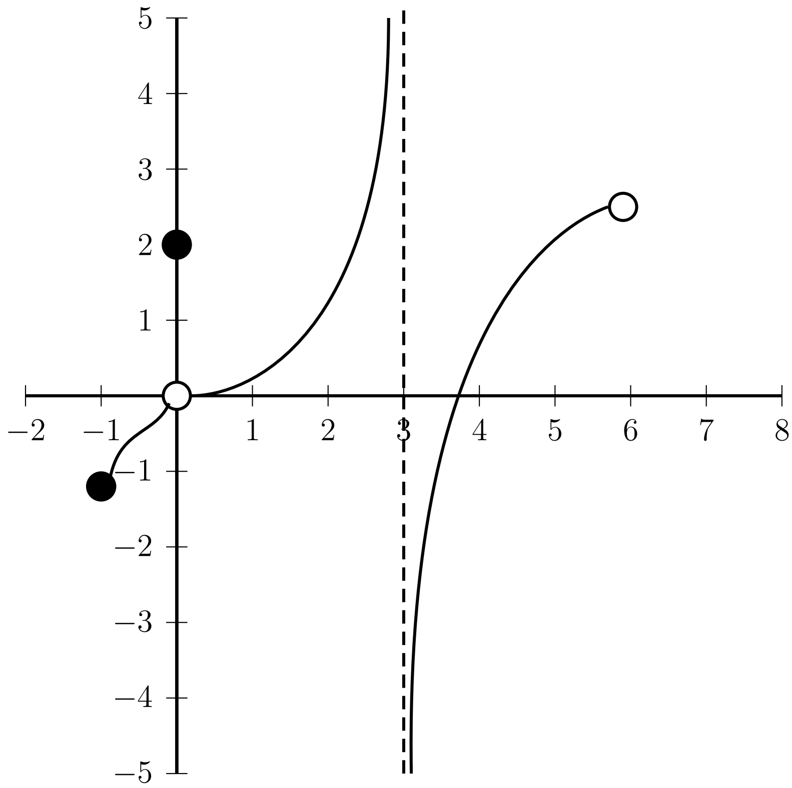
1. Sketch the graph of a function $f(x)$ which satisfies the following properties:
   (a) Domain is $[-1, 3) \cup (3, 6)$
   (b) $\lim_{x \to 3^-} f(x) = +\infty$
   (c) $\lim_{x \to 3^+} f(x) = -\infty$
   (d) $f(0) = 2$
   (e) $\lim_{x \to 0} f(x) = 0$
   (f) $f(x)$ is continuous everywhere on its domain except at $x = 0$

Answers will vary, here is one example...



2. Compute the following limits:

(a) $\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1}$

$$\begin{aligned}
\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} &= \lim_{x\to 1} \frac{(2x+1)(x-1)}{x-1} \\
&= \lim_{x\to 1} (2x+1) \cdot \frac{(x-1)}{x-1} \\
&= \lim_{x\to 1} (2x+1) \cdot \lim_{x\to 1} \frac{(x-1)}{x-1} \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}$$

(b) $\lim_{h\to 0} \frac{\frac{1}{2h+3} - \frac{1}{3}}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{2h+3} - \frac{1}{3}}{h} &= \lim_{h\to 0} \frac{\frac{3-(2h+3)}{3(2h+3)}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{3h(2h+3)} \\
&= \lim_{h\to 0} \frac{-2}{3(2h+3)} \\
&= -\frac{2}{9}
\end{aligned}$$

(c) $\lim_{h\to 0} \frac{\sqrt{2h+1} - 1}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{2h+1} - 1}{h} &= \lim_{h\to 0} \frac{\sqrt{2h+1} - 1}{h} \cdot \frac{\sqrt{2h+1} + 1}{\sqrt{2h+1} + 1} \\
&= \lim_{h\to 0} \frac{(2h+1) - 1}{h(\sqrt{2h+1} + 1)} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2h+1} + 1)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2h+1} + 1} \\
&= 1
\end{aligned}$$

(d) $\lim_{x\to 2^-} \frac{(x+2)(x+1)}{(x-3)(x-2)}$

Note that if we plug in $x = 2$, we end up with $\frac{12}{0}$, so the limit is going to be either ∞ or $-\infty$. Plugging in numbers close to $x = 2$ but less than 2 gives something close to 5 in the numerator, and in the denominator, something close to -1 times something close 0 but negative, thus we have

$$\lim_{x\to 2^-} \frac{(x+2)(x+1)}{(x-3)(x-2)} = \frac{(\approx 4) \cdot (\approx 3)}{(\approx -1) \cdot \text{small negative number}} = +\infty$$

3. Compute the following derivatives (do not simplify your answer):

(a) $\dfrac{d}{dx}\dfrac{x^2-4x+3}{x^2+2x-1}$

We apply the quotient rule here, $f(x) = x^2 - 4x + 3$, $g(x) = x^2 + 2x - 1$, and $f'(x) = 2x - 4$ and $g'(x) = 2x + 2$ to get

$$\frac{d}{dx}\frac{x^2-4x+3}{x^2+2x-1} = \frac{(2x-4)(x^2+2x-1)-(x^2-4x+3)(2x+2)}{(x^2+2x-1)^2}$$

(b) $\dfrac{d}{dx}\sqrt{\dfrac{5x+1}{2x+6}}$

We apply the chain rule, with $f(x) = \sqrt{x}$ and $g(x) = \dfrac{5x+1}{2x+6}$. This gives $f'(x) = \dfrac{1}{2\sqrt{x}}$, and we have to use the quotient rule on $g(x)$:

$$g'(x) = \frac{5(2x+6)-(5x+1)2}{(2x+6)^2}$$

So now we put it all together:

$$\frac{d}{dx}\sqrt{\frac{5x+1}{2x+6}} = \frac{1}{2\sqrt{\frac{5x+1}{2x+6}}}\cdot\frac{5(2x+6)-(5x+1)2}{(2x+6)^2}$$

(c) $\dfrac{d^3}{dx^3}\left(x^2-\dfrac{3}{x}+\dfrac{2}{x^2}\right)$

We need to take three derivatives, so we start with the first:

$$\frac{d}{dx}\left(x^2-\frac{3}{x}+\frac{2}{x^2}\right) = 2x+\frac{3}{x^2}-\frac{4}{x^3}$$

Next we take another derivative:

$$\frac{d}{dx}\left(2x+\frac{3}{x^2}-\frac{4}{x^3}\right) = 2-\frac{6}{x^3}+\frac{12}{x^4}$$

One final derivative gives

$$\frac{d^3}{dx^3}\left(x^2-\frac{3}{x}+\frac{2}{x^2}\right) = \frac{18}{x^4}-\frac{48}{x^5}$$

4. Find the equation of the tangent line to $f(x) = x^2 - 3x + \dfrac{4}{x} - 2$ at $x = 1$.

We use the point-slope form of a line: $y - y_0 = m(x - x0)$, with $y_0 = f(x_0)$ and $m = f'(x_0)$. First, we find $f(x_0) = f(1) = 0$, and then we compute $f'(x)$:

$$f'(x) = 2x - 3 - \frac{4}{x^2}$$

Next, using $f'(x)$ compute above, we find $f'(x_0) = f'(1) = -5$. So we get the equation of the tangent line is:

$$y = -5(x-1)$$