Math 2143 - Brief Calculus with Applications

Exam #2 - 2014.11.06

Solutions

1. Sketch the graph of

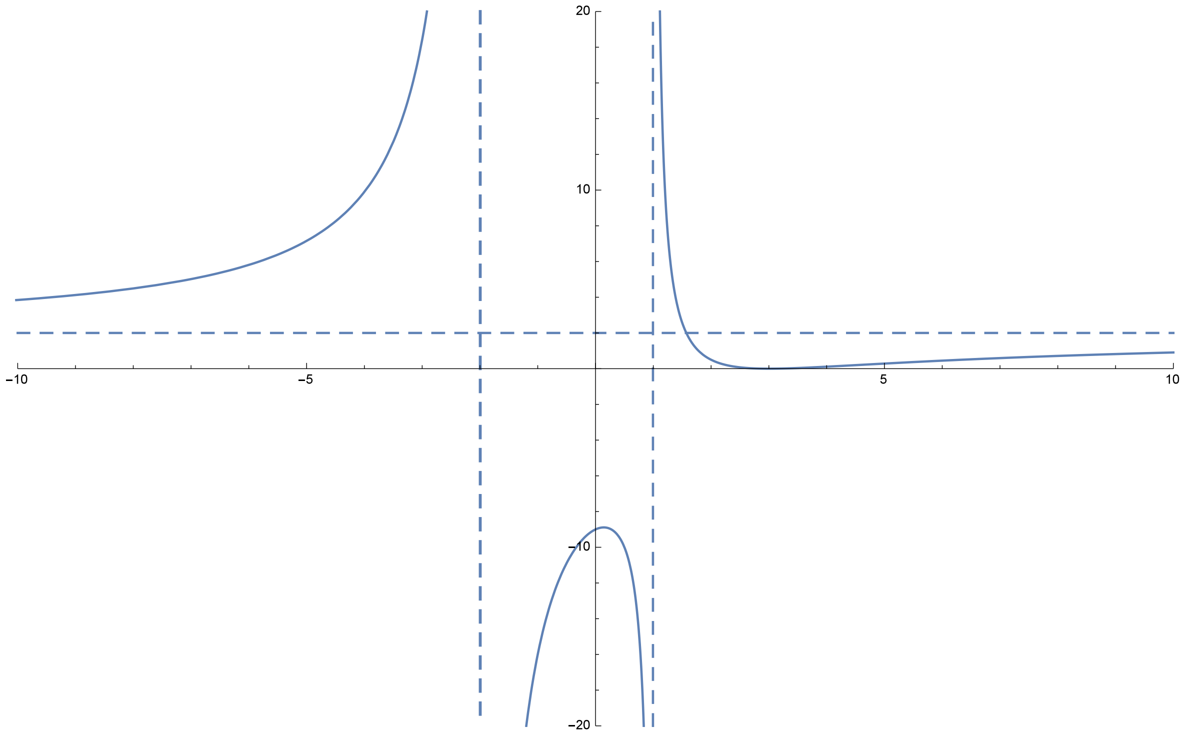
$$f(x) = \frac{2(x-3)^2}{(x+2)(x-1)}$$

without taking derivatives. (Thus, find domain, horizontal and vertical asymptotes, and intercepts).

First, the domain is all real numbers except $x = 1$ and $x = -2$, which are then are vertical asymptotes. To determine how our function behaves at those vertical asymptotes, we compute the following limits:

$$\lim_{x\to -2^-} f(x) = +\infty, \quad \lim_{x\to -2^+} f(x) = -\infty, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = +\infty,$$

The horizontal asymptote is $y = 2$, and the only root is at $x = 3$. Lastly, the $y-$intercept is $y = -9$.



2. Sketch the graph of

$$g(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x - 1$$

by finding the y-intercept, critical points, intervals of increase and decrease, inflection points, and intervals of concavity.

First, the domain is all real numbers since $g(x)$ is a polynomial. The $y-$ intercept is $y = -1$. So next we take a derivative:

$$g'(x) = x^2 - x - 2 = (x-2)(x+1)$$

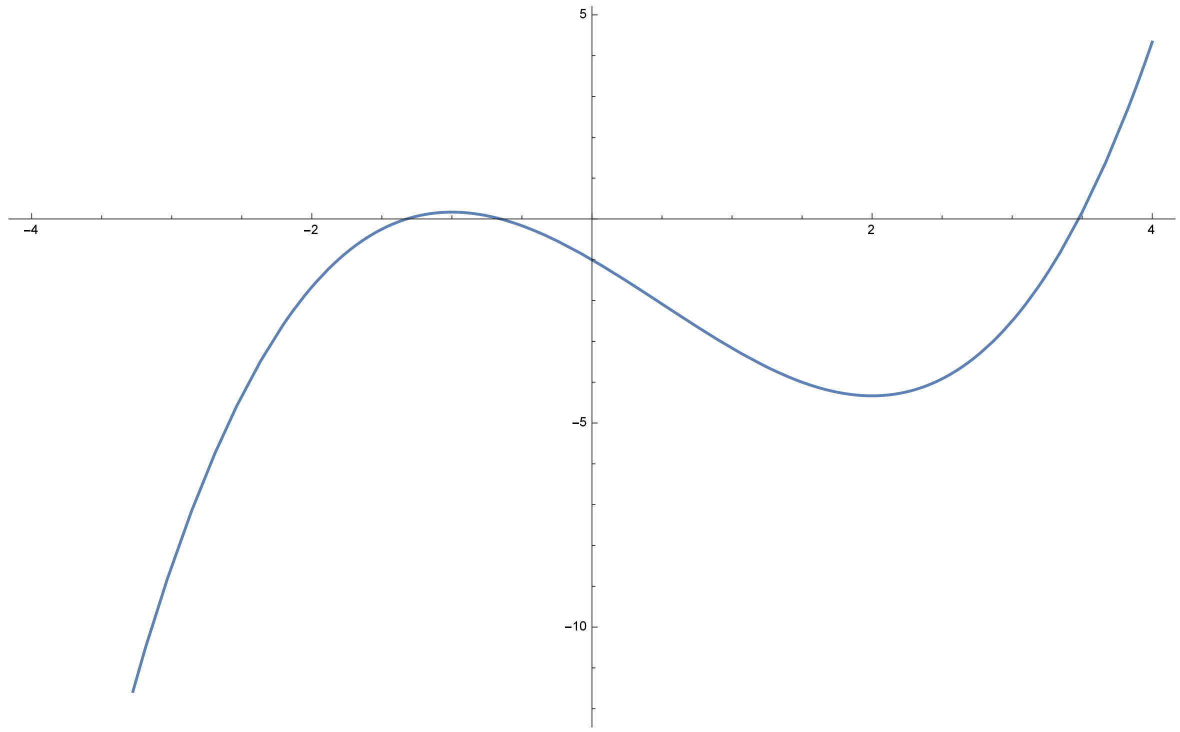
So critical points are at $x = 2$ and $x = -1$. Next, $g'(x) > 0$ for $x < -1$, $g'(x) < 0$ for $-1 < x < 2$ and $g'(x) > 0$ for $x > 2$. So our intervals of increase are $(-\infty, -1) \cup (2, \infty)$. The only interval of decrease is therefore on $(-1, 2)$. We have that $x = -1$ is a local max, and $x = 2$ is a local min, with $g(-1) = \frac{1}{6}$ and $g(2) = -\frac{13}{3}$.

Next, we compute the second derivative:

$$g''(x) = 2x - 1$$

which means that the only possible inflection point occurs at $x = \frac{1}{2}$. If $x < \frac{1}{2}$, then $g''(x) < 0$, and if $x > \frac{1}{2}$, $g''(x) > 0$, therefore $x = \frac{1}{2}$ is an inflection point, and $g(x)$ is concave down on $(-\infty, \frac{1}{2})$ and concave up on $(\frac{1}{2}, \infty)$. Also, $g\left(\frac{1}{2}\right) = -\frac{25}{12}$.

Putting all of this together yields the graph below:



3. Find the absolute maximum and minimum of the function

$$h(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 + 5$$

on the interval $[-3, 2]$.

First we compute a derivative to locate any critical points:

$$h'(x) = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x-1)(x+2)$$

So critical points are at $x = 0$, $x = -2$, and $x = 1$. Now we evaluate the function at the end points and the critical points:

$$h(-3) = \frac{29}{4}, \quad h(-2) = \frac{7}{3}, \quad h(0) = 5, \quad h(1) = \frac{55}{12}, \quad h(2) = \frac{23}{3}$$

With a little bit of head scratching to compare, we see that the smallest value in this list is $\frac{7}{3}$ corresponding to $x = -2$, and the largest, just barely, is at $x = 2$, with a value of $\frac{23}{3}$.

4. Approximate $\sqrt[3]{26}$ as a fraction using differentials.

If we consider the function $f(x) = \sqrt[3]{x}$ with $x_0 = 27$ and $f(x_0) = 3$, we can use the tangent line at $x_0 = 27$ to approximate $\sqrt[3]{26}$. So

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3}\left(\frac{1}{\sqrt[3]{x}}\right)^2$$

and thus

$$f'(27) = \frac{1}{3}\left(\frac{1}{3}\right)^2 = \frac{1}{27}$$

So our tangent line equation looks like:

$$L(x) = \frac{1}{27}(x - 27) + 3$$

Plugging in $x = 26$ gives

$$L(26) = -\frac{1}{27} + 3 = \frac{80}{27}$$

If we convert this result to a decimal, we get $\frac{80}{27} = 2.9629\overline{629}$, with $\sqrt[3]{26} \approx 2.962496$.

5. If $\sqrt{x} + \sqrt{y} = 3$, compute $\frac{dy}{dx}$.

Here we implicitly differentiate, treating $y = y(x)$.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

6. You are pouring paint into a puddle on the floor, forming a perfect circle while doing so. When the radius of this paint circle is 2 feet, you calculate that the change in area is $\frac{1}{4}\pi$ feet^2 per second. At what rate is the radius increasing in this instant?

We use the formula $A = \pi r^2$, with $A = A(t)$ and $r = r(t)$ and differentiate with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We need $\frac{dr}{dt}$, and we are given $\frac{dA}{dt}$ and r. Solving for $\frac{dr}{dt}$ gives

$$\frac{dr}{dt} = \frac{dA/dt}{2\pi r}$$

Setting $\frac{dA}{dt} = \frac{1}{4}\pi$ feet^2 per second and $r = 2$ feet, we get

$$\frac{dr}{dt} = \frac{\pi/4}{4\pi} \text{ feet per second} = \frac{1}{16} \text{ feet per second}$$