Math 2215 - Calculus 1

Exam #1 - 2016.08.29

Solutions

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1. Compute the following limit:

$$\lim_{x\to-\infty} \frac{x^3-3x^2+2}{\sqrt{4x^6-2x^3+5x}}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^3-3x^2+2}{\sqrt{4x^6-2x^3+5x}} &= \lim_{x\to-\infty} \frac{x^3-3x^2+2}{\sqrt{x^6\left(4-\frac{2}{x^3}+\frac{5}{x^5}\right)}} \\
&= \lim_{x\to-\infty} \frac{x^3-3x^2+2}{|x^3|\sqrt{4-\frac{2}{x^3}+\frac{5}{x^5}}} \\
&= \lim_{x\to-\infty} \frac{x^3\left(1-\frac{3}{x}+\frac{2}{x^3}\right)}{|x^3|\sqrt{4-\frac{2}{x^3}+\frac{5}{x^5}}} \\
&= \lim_{x\to-\infty} \frac{x^3}{|x^3|} \cdot \frac{\left(1-\frac{3}{x}+\frac{2}{x^3}\right)}{\sqrt{4-\frac{2}{x^3}+\frac{5}{x^5}}} \\
&= \lim_{x\to-\infty} \frac{x^3}{|x^3|} \cdot \lim_{x\to-\infty} \frac{\left(1-\frac{3}{x}+\frac{2}{x^3}\right)}{\sqrt{4-\frac{2}{x^3}+\frac{5}{x^5}}} \\
&= -1 \cdot \frac{1}{\sqrt{4}} \\
&= -\frac{1}{2}
\end{aligned}$$

2. Compute the following limit:

$$\lim_{x\to3^-} \frac{(x+4)^2(x+1)(x+3)}{(x-1)(x+1)(x-3)^2(x-2)}$$

If we plug in 3, we get something nonzero over something zero. Thus, the limit will be $\pm\infty$. To determine which, we plug in values close to 3 but slightly less than 3. Note that near $x = 3$, the numerator will always be positive, so it is sufficient to look at the denominator. The signs of the terms in the denominator are +, +, + (due to the square on the $(x-3)$ term), and +. Thus, we can conclude that

$$\lim_{x\to3^-} \frac{(x+4)^2(x+1)(x+3)}{(x-1)(x+1)(x-3)^2(x-2)} = +\infty.$$

3. Compute the following limit:

$$\lim_{x\to0} \frac{\sin(3x^2)}{5x^2}$$

$$\begin{aligned}
\lim_{x\to0} \frac{\sin(3x^2)}{5x^2} &= \frac{3}{5} \cdot \lim_{x\to0} \frac{\sin(3x^2)}{3x^2} \\
&= \frac{3}{5} \cdot 1 \\
&= \frac{3}{5}
\end{aligned}$$

4. Compute the following limit:

$$\lim_{h\to0} \frac{(3+h)^3-27}{h}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{(3+h)^3-27}{h} &= \lim_{h\to 0}\frac{(27+27h+9h^2+h^3)-27}{h} \\
&= \lim_{h\to 0}\frac{27h+9h^2+h^3}{h} \\
&= \lim_{h\to 0}\frac{h}{h}\frac{27+9h+h^2}{1} \\
&= \lim_{h\to 0}\frac{h}{h}\cdot\lim_{h\to 0}(27+9h+h^2) \\
&= 1\cdot 27 \\
&= 27
\end{aligned}$$

5. State the algebraic definition of a function $f(x)$ being continuous at the point $x = a$.

The function $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

6. Find the value of a for which the following function is continuous everywhere.

$$f(x) = \begin{cases} 3x^2 - 2x + 1, & x \le 0 \\ 3 - a\cos(2x), & x > 0 \end{cases}$$

Clearly the function is continuous for all $x \neq 0$, so we simply need to make sure that $f(x)$ is continuous at $x = 0$. Using the definition of continuity from the previous problem, we have that $f(0) = 1$, and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x^2 - 2x + 1 = 1$$

as well as,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} 3 - a\cos(2x) = 3 - a$$

For the limit to exist, the limit from the left must equal the limit from the right. Thus $1 = 3 - a$ or $a = 2$. Furthermore, the limit from the left and from the right equal the function value, therefore, if $a = 2$, the function is continuous at $x = 0$, and therefore continuous everywhere.

7. Compute the following limit:

$$\lim_{z\to\infty} \tan\left(\frac{\pi z^2 - 3z + 2}{4z^2 + 2z - 1}\right)$$

$$\begin{aligned}
\lim_{z\to\infty} \tan\left(\frac{\pi z^2 - 3z + 2}{4z^2 + 2z - 1}\right) &= \tan\left(\lim_{z\to\infty}\frac{\pi z^2 - 3z + 2}{4z^2 + 2z - 1}\right) \\
&= \tan\left(\frac{\pi}{4}\right) \\
&= 1
\end{aligned}$$

8. Compute the following limit:

$$\lim_{x\to 0^+}\frac{|5x| + 2x}{|2x| - 5x}$$

Since $x \to 0^+$, this means $x > 0$, therefore

$$\begin{aligned}
\lim_{x\to 0^+}\frac{|5x| + 2x}{|2x| - 5x} &= \lim_{x\to 0^+}\frac{5x + 2x}{2x - 5x} \\
&= \lim_{x\to 0^+}\frac{7x}{-3x} \\
&= -\frac{7}{3}
\end{aligned}$$