Math 2215 - Calculus 1

Exam #3 - 2016.10.10

Solutions

1. Approximate $(1.15)^{5/3}$.

We will linearize the function $f(x) = x^{5/3}$ at $x_0 = 1$. First we take a derivative,

$$f'(x) = \frac{5}{3}x^{2/3}$$

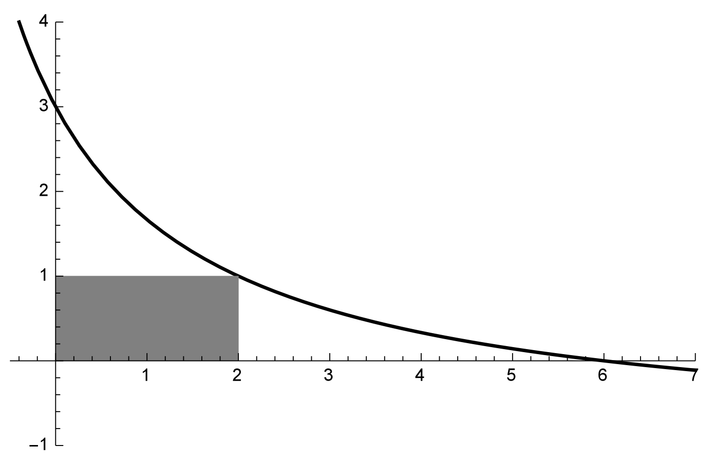
and the note that $f(1) = 1$ and $f'(x) = \frac{5}{3}$. The equation of the tangent line is thus given by

$$y = \frac{5}{3}(x-1) + 1.$$

Setting $x = 1.15$ gives

$$\begin{aligned}(1.15)^{5/3} &\approx \frac{5}{3}(1.15-1)+1\\ &\approx \frac{5}{3}(0.15)+1\\ &\approx \frac{5}{3}\frac{15}{100}+1\\ &\approx \frac{1}{4}+1\\ &\approx \frac{5}{4}.\end{aligned}$$

2. Find the area of the largest rectangle which can be inscribed in the region bounded by the curve $y = \dfrac{6-x}{2+x}$ and the coordinate axis.



The area of the rectangle is given by $A = xy$, and given x, we have $y = \dfrac{6-x}{2+x}$, so

$$A(x) = x\frac{6-x}{2+x} = \frac{6x-x^2}{2+x},$$

for $x \in [0, 6]$. Since this is a continuous function on the closed, bounded domain, we are guaranteed that $A(x)$ has both a absolute max and absolute min on the interval. So we take a derivative:

$$\begin{aligned} A'(x) &= \frac{(6-2x)(2+x)-(6x-x^2)}{(2+x)^2} \\ &= -\frac{x^2+4x-12}{(x+2)^2} \end{aligned}$$

The denominator is never zero, so we simply look at zeros of the numerator:

$$x^2 + 4x - 12 = (x+6)(x-2) \to x = 2, x = -6.$$

Only $x = 2$ is in the interval $[0, 6]$. So we evaluate $A(x)$ at our critical point and the endpoints. Note that $A(0) = A(6) = 0$, so $x = 2$ must be our global max on the interval. To find the area, we simply compute $A(2) = 2$.

3. Find the intervals of increase and decrease, and classify all critical points for the function $f(x) = 2\sin(x) - \cos^2(x)$ on the interval $[0, 2\pi]$.

First we take a derivative:

$$\begin{aligned} f'(x) &= 2\cos(x) + 2\cos(x)\sin(x) \\ &= 2\cos(x)\,(1+\sin(x)) \end{aligned}$$

Setting $f'(x) = 0$ gives $\cos(x) = 0$ or $\sin(x) = -1$.

$$\cos(x) = 0 \to x = \frac{\pi}{2}, \frac{3\pi}{2}$$

and

$$\sin(x) = -1 \to \frac{3\pi}{2}.$$

We have broken up our interval $[0, 2\pi]$ into three sub-intervals: $(0, \pi/2)$, $(\pi/2, 3\pi/2)$, $(3\pi/2, 2\pi)$. First, we plug in a value close to zero in the derivative, gives a positive result. Next we plug in $x = \pi$, which gives $f'(\pi) = -2 < 0$. Lastly, we plug in a value z close to $x = 2\pi$ but slightly less than 2π gives $f'(z) > 0$.

So the intervals of increase are $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The interval of decrease is $(\pi/2, 3\pi/2)$. This implies that the critical point at $x = \pi/2$ is a local max, and the critical point at $x = 3\pi/2$ is a local min.

4. Sketch the graph of the function $f(x) = \dfrac{x}{x^2+1}$. In doing so, compute all asymptotes, intercepts, the first derivative, critical points, intervals of increase and decrease, the second derivative, intervals of concavity, inflection points, local max and mins, as well as global max and mins.

(1) First, there are no vertical asymptotes because the denominator is never zero.
(2) The horizontal asymptote is $y = 0$ since the degree of the numerator is less than that of the denominator.
(3) The graph goes through the origin, which is an $x-$ and $y-$intercept. There are no other $x-$intercepts.
(4) We take a derivative:

$$\begin{aligned} f'(x) &= \frac{(x^2+1) - x(2x)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

(5) Critical points only occur when the numerator is zero since the denominator is never zero. The numerator is zero at $x = \pm 1$.

(6) Testing for intervals of increase and decrease requires inspection of the numerator only, since the denominator is always positive. The numerator is a parabola opening down with roots at $x = \pm 1$. So the derivative is negative on the intervals $(-\infty, -1)$ and $(1, \infty)$, which means the function is decreasing on these two intervals. Similarly, the $f(x)$ is increasing on $(-1, 1)$.

(7) We take a second derivative

$$\begin{aligned} f''(x) &= \frac{d}{dx}\frac{1-x^2}{(x^2+1)^2} \\ &= \frac{-2x(x^2+1)^2-(1-x^2)\left[2(x^2+1)2x\right]}{(x^2+1)^4} \\ &= \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3} \end{aligned}$$

(8) Setting the second derivative to zero gives $x = 0$ or $x = \pm\sqrt{3}$. So we have four intervals of concavity to consider: $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$, $(\sqrt{3}, \infty)$. We get that $f(x)$ is concave down on $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up on $(\sqrt{3}, \infty)$.

(9) We arrive at the conclusion that $x = 0$ and $x = \pm\sqrt{3}$ are inflection points from the intervals of concavity just computed.

(10) By using either the first or second derivative test, we see that there is a local min at $x = -1$ and a local max at $x = 1$. Note that $f(-1) = -1/2$, $f(1) = 1/2$. These are also the global max and mins since we have a horizontal asymptote of $y = 0$.