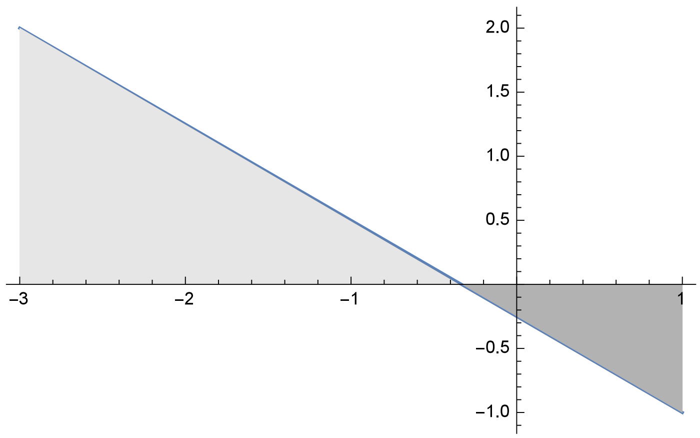
Math 2215 - Calculus 1

Exam #4 - 2016.10.27

Solutions

---

1. Consider the function $f(x) = -\frac{3}{4}x - \frac{1}{4}$ on the interval $[-3, 1]$. Graph this function and geometrically compute the signed area between the function and the x-axis on the interval $[-3, 1]$.



The area of the left (upper) triangle is $A_1 = 1/2 \cdot 2 \cdot 8/3 = 8/3$. The area of the lower right triangle is $A_2 = 1/2 \cdot 4/3 \cdot 1 = 2/3$. The total area is $A_1 - A_2 = 8/3 - 2/3 = 2$.

2. For the function $f(x) = -\frac{3}{4}x - \frac{1}{4}$ on the interval $[-3, 1]$ as given in problem 1, evaluate $A(n)$ (the Riemann sum approximation using n rectangles) by using right endpoints for $f(x)$ on $[-3, 1]$. Simplify fully, and then take $\lim_{n\to\infty} A(n)$, verifying your result with that of problem 1.

First, we compute $\Delta x = (1 - (-3))/n = 4/n$, and the right endpoint formula is $x_k = -3 + k\Delta x = -3 + 4k/n$. So we have

$$\begin{aligned}
A(n) &= \sum_{k=1}^{n} f(x_k)\Delta x \\
&= \sum_{k=1}^{n} \left(-\frac{3}{4}x_k - \frac{1}{4}\right)\frac{4}{n} \\
&= \sum_{k=1}^{n} \left(-\frac{3}{4}\left(-3 + 4\frac{k}{n}\right) - \frac{1}{4}\right)\frac{4}{n} \\
&= \sum_{k=1}^{n} \left(\frac{9}{4} - \frac{3}{n}k - \frac{1}{4}\right)\frac{4}{n} \\
&= \frac{4}{n}\sum_{k=1}^{n} \left(2 - \frac{3}{n}k\right) \\
&= \frac{4}{n}\left(2n - \frac{3}{n}\frac{n(n+1)}{2}\right) \\
&= 8 - 6\frac{n(n+1)}{n^2}
\end{aligned}$$

Taking the limit, we have

$$\begin{aligned} A &= \lim_{n\to\infty} A(n) \\ &= \lim_{n\to\infty} 8 - 6\frac{n(n+1)}{n^2} \\ &= 8 - 6 = 2 \end{aligned}$$

3. Use the Fundamental Theorem of Calculus to compute the following definite integral, verifying your results with those of problems 1 and 2.

$$\int_{-3}^{1} -\frac{3}{4}x - \frac{1}{4}\,\mathrm{d}x$$

$$\begin{aligned} \int_{-3}^{1} -\frac{3}{4}x - \frac{1}{4}\,\mathrm{d}x &= -\frac{3}{8}x^2 - \frac{1}{4}x\Big|_{-3}^{1} \\ &= \left(-\frac{3}{8} - \frac{1}{4}\right) - \left(-\frac{27}{8} + \frac{3}{4}\right) \\ &= -\frac{3}{8} - \frac{2}{8} + \frac{27}{8} - \frac{6}{8} = 2 \end{aligned}$$

4. Compute the average value of $f(x) = -\frac{3}{4}x - \frac{1}{4}$ on the interval $[-3, 1]$.

The average value of $f(x)$ on the interval $[-3, 1]$ is given by

$$\mathrm{Av}(f(x)) = \frac{1}{1-(-3)}\int_{-3}^{1} -\frac{3}{4}x - \frac{1}{4}\,\mathrm{d}x = \frac{1}{4}\cdot 2 = \frac{1}{2}$$

5. Compute the following definite integral:

$$\int_{-\pi}^{\pi} \theta^3\cos(\theta) - 3\theta^2\sin(\theta) + \theta\,\mathrm{d}\theta$$

Since the integrand is odd over an even interval, we can skip the Fundamental Theorem of Calculus and state:

$$\int_{-\pi}^{\pi} \theta^3\cos(\theta) - 3\theta^2\sin(\theta) + \theta\,\mathrm{d}\theta = 0.$$

6. Compute the following indefinite integral:

$$\int \cos^3(4z)\sin(4z)\,\mathrm{d}z$$

We perform a substitution here, let $u = \cos(4z)$, then $\mathrm{d}u = -4\sin(4z)\mathrm{d}z$.

$$\begin{aligned} \int \cos^3(4z)\sin(4z)\,\mathrm{d}z &= -\frac{1}{4}\int u^3\,\mathrm{d}u \\ &= -\frac{1}{16}u^4 + C \\ &= -\frac{1}{16}\cos^4(4z) + C \end{aligned}$$

7. Compute the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}w}\int_{2w}^{3w^2-2w+1} \tan(t+1)\,\mathrm{d}t$$

We use the Fundamental Theorem of Calculus first, and let $F(t)$ be the function such that $F'(t) = \tan(t+1)$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}w}\int_{2w}^{3w^2-2w+1} \tan(t+1)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}w}\left(F(3w^2-2w+1) - F(2w)\right) \\ &= F'(3w^2-2w+1)\cdot(6w-2) - F'(2w)\cdot 2 \\ &= \tan(3w^2-2w+2)\cdot(6w-2) - \tan(2w)\cdot 2 \end{aligned}$$

8. Compute the following indefinite integral:

$$\int \sqrt[3]{5y+1}\,\mathrm{d}y$$

Once again, a substitution: Let $v = 5y+1$, then $\mathrm{d}v = 5\mathrm{d}y$, or $\mathrm{d}y = \frac{1}{5}\mathrm{d}v$:

$$\begin{aligned}\int \sqrt[3]{5y+1}\,\mathrm{d}y &= \int \frac{1}{5}\sqrt[3]{v}\,\mathrm{d}v \\ &= \int \frac{1}{5}v^{1/3}\,\mathrm{d}v \\ &= \frac{1}{5}\cdot\frac{3}{4}v^{4/3} + C \\ &= \frac{3}{20}\sqrt[3]{v^4} + C \\ &= \frac{3}{20}\sqrt[3]{(5y+1)^4} + C\end{aligned}$$

9. Compute the following definite integral:

$$\int_0^{\pi/2} \sin(2\theta) - \cos(4\theta)\,\mathrm{d}\theta$$

$$\begin{aligned}\int_0^{\pi/2} \sin(2\theta) - \cos(4\theta)\,\mathrm{d}\theta &= -\frac{1}{2}\cos(2\theta) - \frac{1}{4}\sin(4\theta)\Big|_0^{\pi/2} \\ &= \frac{1}{2}\cos(2\theta) + \frac{1}{4}\sin(4\theta)\Big|_{\pi/2}^{0} \\ &= \left(\frac{1}{2}\cos(0) + \frac{1}{4}\sin(0)\right) - \left(\frac{1}{2}\cos(\pi) + \frac{1}{4}\sin(2\pi)\right) \\ &= \left(\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 0\right) - \left(\frac{1}{2}\cdot(-1) + \frac{1}{4}\cdot 0\right) \\ &= 1\end{aligned}$$

10. Write the following expression as as *single* definite integral:

$$\int_2^1 f(x)\,\mathrm{d}x + \int_1^5 f(x)\,\mathrm{d}x + \int_5^3 f(x)\,\mathrm{d}x$$

$$\begin{aligned}\int_2^1 f(x)\,\mathrm{d}x + \int_1^5 f(x)\,\mathrm{d}x + \int_5^3 f(x)\,\mathrm{d}x &= \int_1^5 f(x)\,\mathrm{d}x - \int_1^2 f(x)\,\mathrm{d}x - \int_3^5 f(x)\,\mathrm{d}x \\ &= \int_2^3 f(x)\,\mathrm{d}x\end{aligned}$$