Math 2215 - Calculus 1

Exam #5 - 2016.11.14

Solutions

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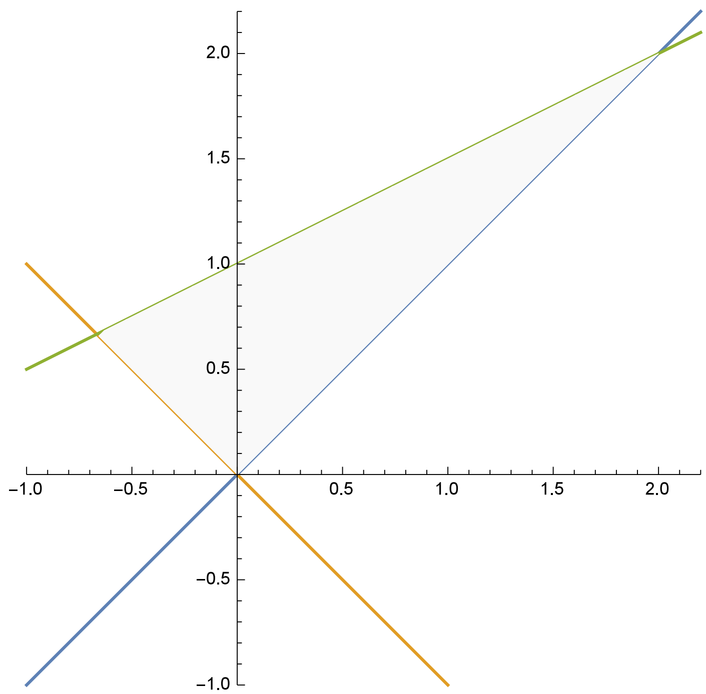
1. Compute the area bounded by the three lines $L_1 = \frac{1}{2}x + 1$, $L_2 = x$ and $L_3 = -x$.



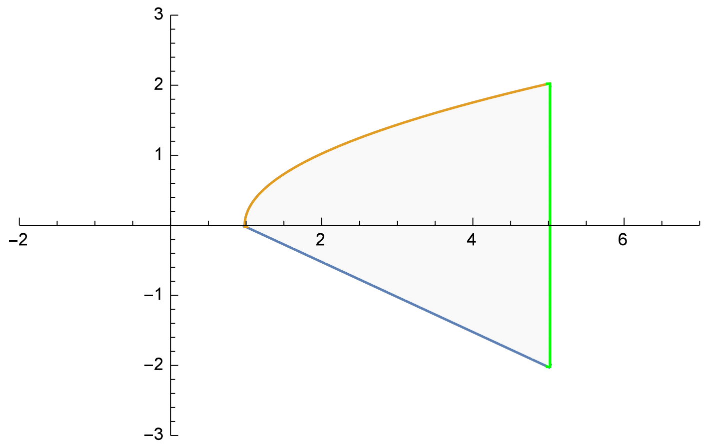
The lines L_1 and L_2 intersect at $x = -2/3$, and L_2 and L_3 intersect at $x = 0$, lastly the lines L_1 and L_3 intersect at $x = 2$. Thus the area can be written as the sum of two integrals:

$$\begin{aligned} A &= \int_{-2/3}^{0} \left(\frac{1}{2}x + 1\right) - (-x)\, \mathrm{d}x + \int_{0}^{2} \left(\frac{1}{2}x + 1\right) - (x)\, \mathrm{d}x \\ &= \int_{-2/3}^{0} \frac{3}{2}x + 1\, \mathrm{d}x + \int_{0}^{2} -\frac{1}{2}x + 1\, \mathrm{d}x \\ &= \left(\frac{3}{4}x^2 + x\right)\Big|_{-2/3}^{0} + \left(-\frac{1}{4}x^2 + x\right)\Big|_{0}^{2} \\ &= \frac{1}{3} + 1 = \frac{4}{3} \end{aligned}$$

2. Consider the region bounded by $f(x) = \sqrt{x-1}$, $g(x) = -\frac{1}{2}x + \frac{1}{2}$, and $x = 5$. Set up, *but do not evaluate*, the integrals required to compute the volume of surface of revolution of this region as follows:

First we sketch a graph of the region. Note that $f(x)$ and $g(x)$ intersect at $x = 1$, and the line $x = 5$ intersects $g(x)$ at $(5, -2)$, and $f(x)$ at $(5, 2)$.

(a) about $y = -2$ using the washer/disk method



For the washer/disk method, we will need a dx integral. The outside radius will be from $f(x)$ to the line $y=-2$, which is $r_o = \sqrt{x-1}+2$, while the inner radius will be from $g(x)$ to $y=-2$, which is $r_i = -\frac{1}{2}x+\frac{1}{2}+2$:

$$V = \int_1^5 \pi\left[\left(\sqrt{x-1}+2\right)^2 - \left(-\frac{1}{2}x+\frac{1}{2}+2\right)^2\right]\,\mathrm{d}x$$

(b) about $y=-2$ using the method of cylindrical shells

For the method of cylindrical shells, we will need a dy integral. This volume will have to broken up into two integrals, the first for $y\in[-2,0]$, and the second for $y\in[0,2]$. This is due to the left endpoint of the width changing functions at $y=0$. For $y\in[-2,0]$, the width is given by right-left $x-$coordinates, which is $5-x$, however this is in terms of x, which we need to convert to y, $w(y)=5-(1-2y)$. The radius function is given by top-bottom: $r(y)=y-(-2)$. The radius function does not change for $y\in[0,2]$, and the width function is still defined to be $5-x$, but in this case, we use the top function $y=\sqrt{x-1}$ to solve for x, thus $w(y)=5-(y^2+1)$. So our volume is given by the sum of integrals:

$$V = \int_{-2}^0 2\pi(y+2)(5-(1-2y))\,\mathrm{d}y + \int_0^2 2\pi(y+2)(5-(y^2+1))\,\mathrm{d}y$$

(c) about $x=5$ using the washer/disk method

For $x=5$, there will be no inside radius, and like part (b), this will be a dy integral which must be broken up into $y\in[-2,0]$ and $y\in[0,2]$. For $y\in[-2,0]$, $r(y)=5-(1-2y)$, and for $y\in[0,2]$, $r(y)=5-(y^2+1)$. Note that these are the same width formulas from part (b).

$$V = \int_{-2}^0 \pi(5-(1-2y))^2\,\mathrm{d}y + \int_0^2 \pi(5-(y^2+1))^2\,\mathrm{d}y$$

(d) about $x=5$ using the method of cylindrical shells

The method of cylindrical shells is a single dx integral, with $r(x)=5-x$ and $h(x)=\sqrt{x-1}-\left(-\frac{1}{2}x+\frac{1}{2}\right)$.

$$V = \int_1^5 2\pi(5-x)\left[\sqrt{x-1}-\left(-\frac{1}{2}x+\frac{1}{2}\right)\right]\,\mathrm{d}x$$

(e) about $x=-1$ using the washer/disk method

Similar to parts (b) and (c), we have a dy integral. The outside radius will always be 6, but the inside radius must once again be broken up into two parts, the first for $y\in[-2,0]$ and the second for $y\in[0,2]$. For $y\in[-2,0]$, the inside radius is $r_i(y)=1+(1-2y)$, and for $y\in[0,2]$, the inside radius is $r_i(y)=1+(y^2+1)$.

$$V = \int_{-2}^0 \pi\left[6^2-(1+(1-2y))^2\right]\,\mathrm{d}y + \int_0^2 \pi\left[6^2-\left(1+(y^2+1)\right)^2\right]\,\mathrm{d}y$$

(f) about $x = -1$ using the method of cylindrical shells

This problem is similar to part (d), and we have a single dx integral, with $r(x) = x + 1$ and $h(x) = \sqrt{x-1} - \left(-\frac{1}{2}x + \frac{1}{2}\right)$.

$$V = \int_1^5 2\pi(x+1)\left[\sqrt{x-1} - \left(-\frac{1}{2}x + \frac{1}{2}\right)\right] \, \mathrm{d}x$$

3. Compute *exactly* the arclength of $f(x) = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2}$ for $x \in [1, 4]$.

First, we compute $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\left(\frac{x}{2}\right)^2 + \frac{1}{2x^2}\right) \\ &= 4\left(\frac{x}{2}\right)^3 \cdot \frac{1}{2} + \frac{1}{2} \cdot (-2) \cdot x^{-3} \\ &= \frac{1}{4}x^3 - x^{-3} \end{aligned}$$

Next, we evaluate $1 + (f'(x))^2$:

$$\begin{aligned} 1 + (f'(x))^2 &= 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 \\ &= 1 + \frac{1}{16}x^6 + \frac{1}{x^6} - \frac{1}{2} \\ &= \frac{1}{16}x^6 + \frac{1}{x^6} + \frac{1}{2} \\ &= \left(\frac{1}{4}x^3 + x^{-3}\right)^2 \end{aligned}$$

Thus, our formula for arclength $\mathcal{L}$ is:

$$\begin{aligned} \mathcal{L} &= \int_1^4 \sqrt{1 + (f'(x))^2} \, \mathrm{d}x \\ &= \int_1^4 \sqrt{\left(\frac{1}{4}x^3 + x^{-3}\right)^2} \, \mathrm{d}x \\ &= \int_1^4 \left|\frac{1}{4}x^3 + x^{-3}\right| \, \mathrm{d}x \\ &= \int_1^4 \frac{1}{4}x^3 + x^{-3} \, \mathrm{d}x \\ &= \frac{1}{16}x^4 - \frac{1}{2x^2}\bigg|_1^4 \\ &= \frac{525}{32} \end{aligned}$$