Math 2215 - Calculus 1

Exam #6 - 2016.11.30

Solutions

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}\text{sech}^{-1}(x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}\text{csch}^{-1}(x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

1. Compute the following integral: $\displaystyle\int \frac{\sin(\ln(2x))}{x}\,\mathrm{d}x$

   We perform a u-substitution: $u = \ln(2x)$, and $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$.

$$\begin{aligned}\int \frac{\sin(\ln(2x))}{x}\,\mathrm{d}x &= \int \sin(u)\,\mathrm{d}u \\ &= -\cos(u) + \mathcal{C} \\ &= -\cos(\ln(2x)) + \mathcal{C}\end{aligned}$$

2. Compute the following derivative: $\dfrac{d}{dx}\sin(x^2)^{\cos(x)+x}$

   First we set $y = \sin(x^2)^{\cos(x)+x}$, and thus $\ln(y) = (\cos(x)+x)\ln(\sin(x^2))$, We can now implicitly differentiate:

$$\begin{aligned}\frac{d}{dx}\ln(y) &= \frac{d}{dx}\left[(\cos(x)+x)\ln(\sin(x^2))\right] \\ \frac{y'}{y} &= (-\sin(x)+1)\ln(\sin(x^2)) + (\cos(x)+x)\frac{\cos(x^2)2x}{\sin(x^2)}\end{aligned}$$

Solving for y' gives

$$\begin{aligned}y' &= \left[(-\sin(x)+1)\ln(\sin(x^2)) + (\cos(x)+x)\frac{\cos(x^2)2x}{\sin(x^2)}\right] y \\ &= \left[(-\sin(x)+1)\ln(\sin(x^2)) + (\cos(x)+x)\frac{\cos(x^2)2x}{\sin(x^2)}\right] \sin(x^2)^{\cos(x)+x}\end{aligned}$$

3. Derive the formula for $\dfrac{d}{dx}\tan^{-1}(x)$ by the method of implicit differentiation.

   Setting $y = \tan^{-1}(x)$ gives $\tan(y) = x$. We then implicitly differentiate:

$$\begin{aligned}\frac{d}{dx}\tan(y) &= \frac{d}{dx}x \\ \sec^2(y)y' &= 1\end{aligned}$$

Solving for y' gives

$$\begin{aligned}y' &= \frac{1}{\sec^2(y)} \\ &= \cos^2(y) \\ &= \cos^2\left(\tan^{-1}(x)\right)\end{aligned}$$

Setting $\theta = \tan^{-1}(x)$, and thus $\tan(\theta) = x$. Drawing a triangle gives opposite side x, adjacent side 1 and hypotenuse $\sqrt{1+x^2}$. Thus $\cos(\theta) = \dfrac{1}{\sqrt{1+x^2}}$ and we finally end up with

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

4. Verify algebraically that $\cosh^2(x) - \sinh^2(x) = 1$.

   First, we rewrite $\sinh(x)$ and $\cosh(x)$ in terms of their exponentials:

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

And then we multiply it all out:

$$\frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{2+2}{4} = 1$$

5. Compute the following integral: $\displaystyle\int 3x\tanh(x^2)\,\mathrm{d}x$

   We perform a u-substitution: $u = x^2$, and $\mathrm{d}u = 2x\,\mathrm{d}x$, and thus $3x\,\mathrm{d}x = \dfrac{3}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int 3x\tanh(x^2)\,\mathrm{d}x &= \frac{3}{2}\int \tanh(u)\,\mathrm{d}u \\ &= \frac{3}{2}\int \frac{\sinh(u)}{\cosh(u)}\,\mathrm{d}u \\ &= \frac{3}{2}\ln\left(|\cosh(u)|\right) + \mathcal{C} \\ &= \frac{3}{2}\ln\left(\left|\cosh(x^2)\right|\right) + \mathcal{C} \\ &= \frac{3}{2}\ln\left(\cosh(x^2)\right) + \mathcal{C}\end{aligned}$$

On the last line above, since $\cosh(z) \geq 1$ for all z, the absolute value can be removed.

6. Compute the following derivative: $\dfrac{d}{dx}\ln\left(\dfrac{(5x-1)^2(3x+2)^5}{(4x-7)^6(3x-2)^7}\right)$.

   First we simplify our function using properties of logarithms:

$$\ln\left(\frac{(5x-1)^2(3x+2)^5}{(4x-7)^6(3x-2)^7}\right) = 2\ln(5x-1) + 5\ln(3x+2) - 6\ln(4x-7) - 7\ln(3x-2)$$

What we now have is easy to differentiate:

$$\begin{aligned}\frac{d}{dx}\ln\left(\frac{(5x-1)^2(3x+2)^5}{(4x-7)^6(3x-2)^7}\right) &= \frac{d}{dx}\left[2\ln(5x-1) + 5\ln(3x+2) - 6\ln(4x-7) - 7\ln(3x-2)\right] \\ &= 2\frac{5}{5x-1} + 5\frac{3}{3x+2} - 6\frac{4}{4x-7} - 7\frac{3}{3x-2}\end{aligned}$$

7. Compute the following integral: $\displaystyle\int \frac{9x}{x^2\sqrt{1-x^4}}\,\mathrm{d}x$

   We perform a u-substitution: $u = x^2$, and $\mathrm{d}u = 2x\,\mathrm{d}x$, and thus $9x\,\mathrm{d}x = \dfrac{9}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{9x}{x^2\sqrt{x^4-1}}\,\mathrm{d}x &= \frac{9}{2}\int \frac{1}{u\sqrt{1-u^2}}\,\mathrm{d}u \\ &= -\frac{9}{2}\mathrm{sech}^{-1}(u) + \mathcal{C} \\ &= -\frac{9}{2}\mathrm{sech}^{-1}(x^2) + \mathcal{C}\end{aligned}$$

8. Compute the following derivative: $\frac{d}{dx}\log_7\left(2^{3x}-4^{5x}+1\right)$.

$$\begin{aligned}\frac{d}{dx}\log_7\left(2^{3x}-4^{5x}+1\right) &= \frac{1}{\ln(7)}\frac{d}{dx}\ln\left(2^{3x}-4^{5x}+1\right)\\ &= \frac{1}{\ln(7)}\frac{\frac{d}{dx}\left(2^{3x}-4^{5x}+1\right)}{2^{3x}-4^{5x}+1}\\ &= \frac{1}{\ln(7)}\frac{\ln(2)2^{3x}\cdot 3-\ln(4)4^{5x}\cdot 5}{2^{3x}-4^{5x}+1}\end{aligned}$$

9. If $f(x)=x^3+3x^2+4x-1$, verify that $f(x)$ is invertible by showing that $f'(x)>0$ for all x. Then compute $\frac{d}{dx}f^{-1}(x)$ at $x=7$.

First, $f'(x)=3x^2+6x+4$, and computing the discriminant gives $d=36-48=-12<0$, thus there are no roots to the derivative and thus $f'(x)>0$ which means the function is always increasing. Thus, $f(x)$ is invertible.

To compute the derivative of the inverse at $x=7$, we need to find $f^{-1}(7)$. By plugging simple small numbers into $f(x)$, we see that $f(1)=7$, thus $f^{-1}(7)=1$. So now we have that

$$\begin{aligned}\left.\frac{d}{dx}f^{-1}(x)\right|_{x=7} &= \frac{1}{f'(f^{-1}(7))}\\ &= \frac{1}{f'(1)}\\ &= \frac{1}{13}\end{aligned}$$