Math 2215 - Calculus 1

Exam #1 - 2017.01.27

Solutions

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1. Compute the following integral: $\int 2z^5 \cos\left(z^2\right)\, \mathrm{d}z$

   We will use substitution here. Setting $u = z^2$, we have $\mathrm{d}u = 2z\,\mathrm{d}z$.

$$\int 2z^5 \cos\left(z^2\right)\, \mathrm{d}z = \int u^2 \cos(u)\,\mathrm{d}u$$

We can do integration by parts twice, but we will instead use tabular integration:

$+$	u^2	$\searrow$	$\cos(u)$
$-$	$2u$	$\searrow$	$\sin(u)$
$+$	2	$\searrow$	$-\cos(u)$
$-$	0	$\searrow$	$-\sin(u)$

$$\int u^2 \cos(u)\,\mathrm{d}u = u^2\sin(u) + 2u\cos(u) - 2\sin(u) + \mathcal{C}$$

Substituting $u = z^2$ back in, we get

$$\int 2z^5 \cos\left(z^2\right)\, \mathrm{d}z = z^4\sin(z^2) + 2z^2\cos(z^2) - 2\sin(z^2) + \mathcal{C}$$

2. Compute the following integral: $\int \frac{x+1}{\sqrt{4+x^2}}\,\mathrm{d}x$

   We break this up into two integrals:

$$\int \frac{x+1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int \frac{x}{\sqrt{4+x^2}}\,\mathrm{d}x + \int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$$

The first integral is straight forward:

$$\int \frac{x}{\sqrt{4+x^2}}\,\mathrm{d}x = \sqrt{4+x^2},$$

and can be found by using a simple substitution of $u = 4 + x^2$ if needed. For the second integral, we perform the substitution $x = 2\tan(\theta)$, with $\mathrm{d}x = 2\sec^2(\theta)$.

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x &= \int \frac{2\sec^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,\mathrm{d}\theta \\ &= \int \frac{\sec^2(\theta)}{\sec(\theta)}\,\mathrm{d}\theta \\ &= \int \sec(\theta)\,\mathrm{d}\theta \\ &= \ln\left(\left|\sec(\theta) + \tan(\theta)\right|\right)\end{aligned}$$

We must convert trig functions in terms of θ back into expressions involving x. If $x = 2\tan(\theta)$, then $\sec(\theta) = \frac{\sqrt{4+x^2}}{2}$, and thus

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \ln\left(\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right|\right)$$

Putting this all together, our final answer is:

$$\int \frac{x+1}{\sqrt{4+x^2}}\,\mathrm{d}x = \sqrt{4+x^2} + \ln\left(\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right|\right) + \mathcal{C}$$

3. Compute the following integral: $\int \frac{\cos(w)}{\sin^2(w)(3+4\sin(w))}\,\mathrm{d}w$

First, we do a substitution of $u = \sin(w)$, and thus $du = \cos(w)\mathrm{d}w$.

$$\int \frac{\cos(w)}{\sin^2(w)(3+4\sin(w))}\,\mathrm{d}w = \int \frac{1}{u^2(3+4u)}\,\mathrm{d}u$$

To do this integral, we do partial fractions:

$$\frac{1}{u^2(3+4u)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{3+4u}/$$

Multiplying through the entire equation by $u^2(3+4u)$ to get

$$1 = Au(3+4u) + B(3+4u) + Cu^2$$

Setting $u = 0$ gives the equation $1 = 3B$, or $B = \frac{1}{3}$. Setting $u = -\frac{3}{4}$ gives $C = \frac{16}{9}$, and for the last unknown, A, we need to pick another value of u, so setting $u = 1$, we end up with $A = -\frac{4}{9}$. So now we have

$$\begin{aligned}\int \frac{1}{u^2(3+4u)}\,\mathrm{d}u &= -\frac{4}{9}\int \frac{1}{u}\,\mathrm{d}u + \frac{1}{3}\int \frac{1}{u^2}\,\mathrm{d}u + \frac{16}{9}\int \frac{1}{3+4u}\,\mathrm{d}u \\ &= -\frac{4}{9}\ln(|u|) - \frac{1}{3}\frac{1}{u} + \frac{4}{9}\ln(|3+4u|) + \mathcal{C} \\ &= -\frac{4}{9}\ln(|\sin(w)|) - \frac{1}{3}\frac{1}{\sin(w)} + \frac{4}{9}\ln(|3+4\sin(w)|) + \mathcal{C}\end{aligned}$$

4. Compute the following limit: $\lim\limits_{x\to 0^+} \tan(x)\ln(x)$

First we note that this is of the form $0 \cdot \infty$, so we need to rewrite it as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Since we will most likely will want to get rid of any $\ln(x)$ in the resulting limit, we will rewrite out limit as

$$\lim_{x\to 0^+} \tan(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)}$$

which is in the form $\frac{\infty}{\infty}$, so we can apply l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2(x)} \\ &= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x} \\ &= -\lim_{x\to 0^+} \sin(x)\cdot\frac{\sin(x)}{x} \\ &= -(0\cdot 1) \\ &= 0\end{aligned}$$

5. Determine the value (if convergent) of the following integral: $\displaystyle\int_{-\infty}^{\infty} \frac{4}{4+r^2}\,\mathrm{d}r$

First, we break this up as

$$\int_{-\infty}^{\infty} \frac{4}{4+r^2}\,\mathrm{d}r = \int_{-\infty}^{0} \frac{4}{4+r^2}\,\mathrm{d}r + \int_{0}^{\infty} \frac{4}{4+r^2}\,\mathrm{d}r$$

We first rewrite the integrand to make it suitable for a substitution:

$$\begin{aligned}\frac{4}{4+r^2} &= \frac{4}{4\left(1+\left(\frac{r}{2}\right)^2\right)} \\ &= \frac{1}{1+\left(\frac{r}{2}\right)^2}\end{aligned}$$

The correct substitution will be $u = \frac{r}{2}$, with $2\mathrm{d}u = \mathrm{d}r$. The limits of each integral remain unchanged, so our new integrals are

$$\begin{aligned}
\int_{-\infty}^{0} \frac{4}{4+r^2}\,\mathrm{d}r + \int_{0}^{\infty} \frac{4}{4+r^2}\,\mathrm{d}r &= 2\int_{-\infty}^{0} \frac{1}{1+u^2}\,\mathrm{d}u + 2\int_{0}^{\infty} \frac{1}{1+u^2}\,\mathrm{d}u \\
&= 2\lim_{A\to-\infty}\int_{A}^{0} \frac{1}{1+u^2}\,\mathrm{d}u + 2\lim_{B\to\infty}\int_{0}^{B} \frac{1}{1+u^2}\,\mathrm{d}u \\
&= 2\lim_{A\to-\infty} \tan^{-1}(u)\Big|_{A}^{0} + 2\lim_{B\to\infty} \tan^{-1}(u)\Big|_{0}^{B} \\
&= 2\lim_{A\to-\infty}\left(\tan^{-1}(0) - \tan^{-1}(A)\right) + 2\lim_{B\to\infty}\left(\tan^{-1}(B) - \tan^{-1}(0)\right) \\
&= 2\left(0 - \left(-\frac{\pi}{2}\right)\right) + 2\left(\frac{\pi}{2} - 0\right) \\
&= \pi + \pi \\
&= 2\pi
\end{aligned}$$

6. Use the comparison test to determine if the following integral is convergent: $\displaystyle\int_0^\infty \frac{\sin^2(x)}{1+e^x}\,\mathrm{d}x$

We have the following string of inequalities:

$$0 \le \frac{\sin^2(x)}{1+e^x} \le \frac{1}{1+e^x} \le \frac{1}{e^x} = e^{-x}$$

We know that the following integral is convergent:

$$\int_0^\infty e^{-x}\,\mathrm{d}x,$$

and thus the integral in question is also convergent.