Math 2215 - Calculus 2

Exam #2 - 2017.02.24

Solutions

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1. Solve the following initial value problem explicitly: $y' = e^{x+y}, \quad y(0) = 3$

First we separate variables:

$$e^{-y}\mathrm{d}y = e^x\mathrm{d}x$$

Integrating with respect to each variable gives

$$-e^{-y} = e^x + \mathcal{C}.$$

Using the initial condition $y(0) = 3$ with the above equation gives $e^{-3} = -1 + \mathcal{C}$, thus $\mathcal{C} = e^{-3} + 1$. Solving for y with the constant now found, gives

$$y = \ln\left(\left|1 + e^{-3} - e^x\right|\right).$$

2. Write the parametric equations $(x, y) = (x(t), y(t))$ which represents a circle of radius 3 centered at $(-1, 1)$ such that all of the following holds simultaneously:
(a) $(x(0), y(0)) = (-1, 4)$,
(b) the curve is oriented counter-clockwise,
(c) it takes a total of π time units to complete a full revolution back to the point $(-1, 4)$.

We know that $(x(t), y(t)) = (\cos(t), \sin(t))$ is the parameterization of the unit circle, oriented counter-clockwise. We give it radius 3 by multiplying the whole equation by 3, and then to center it at $(-1, 1)$, $(x(t), y(t)) = (-1 + 3\cos(t), 1 + 3\sin(t))$. In order to give it period π, we replace t with $2t$: $(x(t), y(t)) = (-1 + 3\cos(2t), 1 + 3\sin(2t))$. We now need to have the curve start at the top of the circle, instead of the right of the circle. To do this, we simply shift the time by $\pi/4$ units:

$$(x(t), y(t)) = \left(-1 + 3\cos\left(2\left(t + \frac{\pi}{4}\right)\right), 1 + 3\sin\left(2\left(t + \frac{\pi}{4}\right)\right)\right)$$

3. Consider the parametric equations $(x(t), y(t)) = (\sqrt{t^2 + 3}, \sin(\pi t))$.

(a) Verify that $(x(-1), y(-1)) = (x(1), y(1))$.

$$(x(-1), y(-1)) = (2, 0) = (x(1), y(1))$$

(b) Find the equations of the two tangents lines to $(x(t), y(t))$ for $t = -1$ and $t = 1$.

First, we compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\pi\cos(\pi t)}{\frac{t}{\sqrt{t^2+3}}} \\ &= \frac{\pi\cos(\pi t)\sqrt{t^2 + 3}}{t}\end{aligned}$$

We next plug in $t = -1$ and $t = 1$ to get

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=-1} = 2\pi, \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = -2\pi,$$

Thus the equations of the tangent lines at $(2, 0)$ are given by (for $t = -1$ and $t = 1$ respectively):

$$y - 0 = 2\pi(x - 2), \quad y - 0 = -2\pi(x - 2).$$

(c) Find all values of t on the interval $[-1, 1]$ for which $(x(t), y(t))$ has a horizontal or vertical tangent line.

To find the vertical tangent lines, we look at where the denominator is zero (and the numerator is not), we see that $t = 0$ is the only value.

To find horizontal tangent lines, we look at the numerator, the only term which can be zero is $\cos(\pi t)$, which happens to be zero at $t = \frac{1}{2}$ and $t = -\frac{1}{2}$.

4. Consider the parametric equations $(x(t), y(t)) = (t\cos(t), t\sin(t))$. Verify that the arclength of the curve for $-1 \le t \le 1$ is given by the integral

$$\int_{-\pi/4}^{\pi/4} \sec^3(\theta)\,\mathrm{d}\theta$$

You do not have to calculate this integral, just simplify your arclength formula until you end up with this integral.

First we compute the derivatives of the components:

$$(x'(t), y'(t)) = (\cos(t) - t\sin(t), \sin(t) + t\cos(t))$$

Squaring each of these:

$$\big((x'(t))^2, (y'(t))^2\big) = \big(\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t), \sin^2(t) + 2t\cos(t)\cos(t) + 2t^2\cos^2(t)\big)$$

Adding and using $\sin^2(t) + \cos^2(t) = 1$ gives

$$(x'(t))^2 + (y'(t))^2 = 1 + t^2.$$

So our integral for arclength is

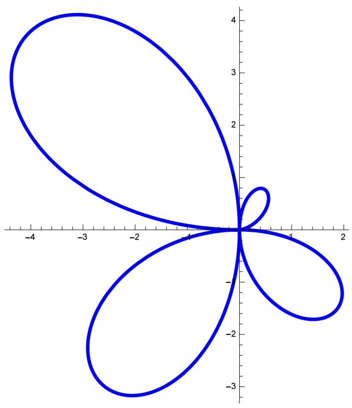
$$\int_{-1}^{1} \sqrt{1 + t^2}\,\mathrm{d}t.$$

To convert to the integral in question, we let $t = \tan(\theta)$, and $\mathrm{d}t = \sec^2(\theta)\mathrm{d}\theta$, where the limits change from -1 to 1, to $-\frac{\pi}{4}$ to $\frac{\pi}{4}$:

$$\begin{aligned}\int_{-1}^{1} \sqrt{1 + t^2}\,\mathrm{d}t &= \int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2(\theta)}\sec^2(\theta)\,\mathrm{d}\theta \\ &= \int_{-\pi/4}^{\pi/4} \sec^3(\theta)\,\mathrm{d}\theta\end{aligned}$$

5. Plot the polar curve $r = \theta\sin(2\theta)$ for $0 \le \theta \le 2\pi$.

First note that $\sin(2\theta) = 0$ for $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$. Furthermore, $\sin(2\theta) > 0$ for $0 < \theta < \pi/2$ and $\pi < \theta < 3\pi/2$, while $\sin(2\theta) < 0$ for $\pi/2 < \theta < \pi$ and $3\pi/2 < \theta < 2\pi$. So we will start our graph in the first quadrant, then the second quadrant graph gets reflected into the fourth quadrant, which then continues into the third quadrant, and finally finishes in the second quadrant (where it ends up at the origin at $\theta = 2\pi$).



6. Compute the area of the loop in the first quadrant of the graph of the polar equation $r = \theta\sin(2\theta)$ from the previous problem. You may need to make use of the trigonometric identity:

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$$

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int_0^{\pi/2} (\theta\sin(2\theta))^2 \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \theta^2\sin^2(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \theta^2(1-\cos(4\theta))\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \theta^2\,\mathrm{d}\theta - \frac{1}{4}\int_0^{\pi/2} \theta^2\cos(4\theta)\,\mathrm{d}\theta
\end{aligned}$$

Obviously we can do the first integral. To do the second we perform tabular integration to get the result:

$+$	θ^2	$\searrow$	$\cos(4\theta)$
$-$	θ	$\searrow$	$\frac{1}{4}\sin(4\theta)$
$+$	2	$\searrow$	$-\frac{1}{16}\cos(4\theta)$
$-$	0	$\searrow$	$-\frac{1}{64}\sin(4\theta)$

$$\begin{aligned}
\mathcal{A} &= \frac{1}{4}\int_0^{\pi/2} \theta^2\,\mathrm{d}\theta - \frac{1}{4}\int_0^{\pi/2} \theta^2\cos(4\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\cdot\frac{1}{3}\theta^3\Big|_0^{\pi/2} - \frac{1}{4}\left[\frac{1}{4}\theta^2\sin(4\theta) + \frac{1}{8}\theta\cos(4\theta) - \frac{1}{32}\sin(4\theta)\right]\Big|_0^{\pi/2} \\
&= \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{\pi^3}{8} - \frac{1}{4}\left[\frac{1}{8}\cdot\frac{\pi}{2}\cos(2\pi)\right] \\
&= \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{\pi^3}{8} - \frac{1}{4}\cdot\frac{\pi}{16} \\
&= \frac{\pi^3}{96} - \frac{\pi}{64}
\end{aligned}$$