Math 2215 - Calculus 2

Exam #3 - 2017.04.20

Solutions

For problems 1 through 5, determine whether each series converges absolutely, conditionally, or is divergent.

1. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\left(\frac{3+2k}{5-3k}\right)^k$

We use the root test:

$$\begin{aligned}\lim_{k\to\infty}|a_k|^{1/k} &= \lim_{k\to\infty}\left|\frac{3+2k}{5-3k}\right| \\ &= \frac{2}{3} < 1\end{aligned}$$

Since the limit is less than 1, the series is absolutely convergent.

2. $\displaystyle\sum_{k=1}^{\infty} 2k^{-8/7}$

This is a simple p-series comparison test, with $p = \frac{8}{7} > 1$, thus the series is absolutely convergent.

3. $\displaystyle\sum_{k=1}^{\infty}\frac{k+\cos(k)}{\sqrt{k^3+k}}$

We can use the comparison test here. If $a_k = \dfrac{k+\cos(k)}{\sqrt{k^3+k}}$, then setting $b_k = k^{-1/2}$, we have that by the limit comparison test

$$\begin{aligned}\lim_{k\to\infty}\frac{a_k}{b_k} &= \lim_{k\to\infty}\frac{k+\cos(k)}{\sqrt{k^3+k}}\cdot\sqrt{k} \\ &= \lim_{k\to\infty}\frac{k^{3/2}+\sqrt{k}\cos(k)}{k^{3/2}} \\ &= 1\end{aligned}$$

Since the p-series with $p = 1/2$ diverges, so does the series in question.

4. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{3+2k}{5-3k}$

Since $\displaystyle\lim_{k\to\infty} a_k \neq 0$, the series diverges.

5. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k}{k^2+1}$

Since this is an alternating series, such that $a_k \to 0$ as $k \to \infty$, the series is at least conditionally convergent. By using the comparison series $1/k$, there is no absolute convergence, thus we only have conditional convergence.

6. Compute, exactly, the following series: $\displaystyle\sum_{k=1}^{\infty}\frac{4}{3k(k+5)}$

To do an exact computation, we rewrite as a telescoping series. To do this, we need to use partial fraction decomposition:

$$\frac{4}{3k(k+5)} = \frac{A}{k+5} + \frac{B}{3k}.$$

This yields the equation

$$4 = A(3k) + B(K+5).$$

Setting $k = 0$ gives $B = 4/5$, and $k = -5$ gives $A = -4/15$. Thus

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{4}{3k(k+5)} &= \sum_{k=1}^{\infty} -\frac{4}{15}\frac{1}{k+5} + \frac{4}{5}\frac{1}{3k} \\ &= \frac{4}{15}\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+5} \\ &= \frac{4}{15}\left[\left(\frac{1}{1} - \frac{1}{6}\right) + \left(\frac{1}{2} - \frac{1}{7}\right) + \left(\frac{1}{3} - \frac{1}{8}\right) + \dots\right]\end{aligned}$$

Note that everything cancels except for the first 6 positive terms, thus:

$$\sum_{k=1}^{\infty} \frac{4}{3k(k+5)} = \frac{4}{15}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right]$$

7. Determine the interval of convergence for the series $\displaystyle\sum_{k=0}^{\infty} \frac{k}{4^k}(x+1)^k$

We apply the ratio test:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{k+1}{4^{k+1}}(x+1)^{k+1} \cdot \frac{4^k}{k}\frac{1}{(x+1)^k}\right| \\ &= \lim_{k\to\infty}\left|\frac{k+1}{k} \cdot \frac{1}{4} \cdot (x+1)\right| \\ &= \left|\frac{x+1}{4}\right| \lim_{k\to\infty}\left|\frac{k+1}{k}\right| \\ &= \left|\frac{x+1}{4}\right|\end{aligned}$$

Requiring that the limit is less than 1 gives the inequality:

$$-1 < \frac{x+1}{4} < 1 \longrightarrow -5 < x < 3.$$

Testing the endpoints, if $x = -5$, then the series reduces to:

$$\sum_{k=0}^{\infty} k \cdot (-1)^k,$$

which diverges. Setting $x = 3$ gives the series

$$\sum_{k=0}^{\infty} k,$$

which also diverges. Thus, the interval of convergence is $(-5, 3)$.

8. Find a power series representation, and interval of convergence, of the function $f(x) = \dfrac{4}{6-3x}$.

$$
\begin{aligned}
f(x) &= \frac{4}{6-3x} \\
&= \frac{4}{6\left(1-\frac{x}{2}\right)} \\
&= \frac{2}{3}\frac{1}{1-\frac{x}{2}} \\
&= \frac{2}{3}\left[1+\frac{x}{2}+\frac{x^2}{2^2}+\frac{x^3}{2^3}+\dots\right] \\
&= \frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k
\end{aligned}
$$

Since this is a geometric series, we require that $|x/2| < 1$, or $-2 < x < 2$. Thus the interval of convergence is $(-2, 2)$.

9. Show that the Maclaurin series for $f(x) = (x+1)\cos(x)$ can be written as follows:

$$
f(x) = (x+1)\cos(x) = 1 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{(2k)!} - \frac{1}{2(k-1)!}\right)x^{2k}
$$

Here you may assume that all series converge for finite x, and may thus be broken up and rearranged as needed.

We start with the Maclaurin series for $\cos(x)$:

$$
\cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}
$$

Then multiply by $x+1$:

$$
\begin{aligned}
(x+1)\cos(x) &= (x+1)\sum_{k=0}^{\infty}\frac{1}{(2k)!}x^{2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(x^{2(k+1)} + x^{2k}\right) \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2(k+1)} + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2(k-1))!}x^{2k} + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \\
&= 1 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2(k-1))!}x^{2k} + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \\
&= 1 + \sum_{k=1}^{\infty}\frac{-1\cdot(-1)^k}{(2(k-1))!}x^{2k} + \frac{(-1)^k}{(2k)!}x^{2k} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{(2k)!} - \frac{1}{2(k-1)!}\right)x^{2k}
\end{aligned}
$$

10. Using the Maclaurin series for $\tan(x)$, compute the following limit, and then verify your work by applying l'Hôpital's rule.

$$
\lim_{x\to 0}\frac{\tan(2x)}{3x}
$$

First, we recognize that $\tan(2x)$ is odd, and thus only odd terms will be in the Maclaurin expansion. Note that $f(0) = 0$, and $f'(x) = 2\sec^2(2x)$, with $f'(0) = 2$. We could compute $f''(x)$, but we know $f''(0) = 0$ since the

Maclaurin series has to be odd. So we now have that

$$\tan(2x) = 2x + cx^3 + \dots,$$

where c is a constant whose value we need not determine. Thus

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(2x)}{3x} &= \lim_{x\to 0} \frac{2x + cx^3 + \dots}{3x} \\
&= \lim_{x\to 0} \frac{2}{3} + \frac{c}{3}x^2 + \dots \\
&= \frac{2}{3} + 0 \\
&= \frac{2}{3}.
\end{aligned}$$

If we apply l'Hôpital's rule (since we have a 0/0 form):

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(2x)}{3x} &= \lim_{x\to 0} \frac{2\sec^2(2x)}{3} \\
&= \frac{2}{3}
\end{aligned}$$