Math 2283 - Introduction to Logic
Final Exam

Assigned: 2017.04.20

Due: 2017.05.04 at 11:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire. Remember, you have two whole weeks to work on this, your masterpiece will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ________________________________

Signature: ________________________________

Date: ________________________________

Definitions:

A relation R is *strongly connected* if and only if $\forall\, x, y,\ xRy \vee yRx$.

A relation S is *antisymmetric* if and only if $\forall\, x, y,\ xRy \wedge yRx \rightarrow x = y$.

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1. (15 pts) Applying the method of truth tables, we may introduce into sentential calculus new terms. We can, for instance, introduce the symbol "Δ":

$$p \,\Delta\, q \overset{def}{\longleftrightarrow} \text{ neither } p \text{ nor } q.$$

Construct the fundamental truth table for this function, which would comply with the intuitive meaning ascribed to the symbol "$\triangle$". Next, using the definition of "Δ" and the corresponding truth table, verify, with the help of truth tables, that the following sentences are true and may be accepted as laws of sentential calculus.

(a) $\sim p \leftrightarrow (\, p \,\Delta\, p\,)$
(b) $(\, p \vee q\,) \leftrightarrow [\,(\, p \,\Delta\, q\,) \,\Delta\, (\, p \,\Delta\, q\,)\,]$
(c) $(\, p \rightarrow q\,) \leftrightarrow [\,[\,(p \,\Delta\, p) \,\Delta\, q\,] \ \Delta\ [\,(\, p \,\Delta\, p) \,\Delta\, q\,]\,]$

2. (10 pts) Prove that if a relation R is antisymmetric, then R' is connected.

3. (10 pts) Prove that if a relation R is reflexive, then R is not asymmetric.

4. (10 pts) Prove that if a relation R' is strongly connected, then R is asymmetric.

5. (20 pts) Let R, S, and T be arbitrary relations. Only one of the the following sentences is true. Determine, with proof, which one is true, and describe what fails in trying to prove the other sentence.

$$(R \cap S)/T \subseteq ((R/T) \cap (S/T)), \quad (R \cap S)/T \supseteq ((R/T) \cap (S/T))$$

6. (15 pts) Prove the following theorem:

$$p \rightarrow [\ (p \rightarrow q) \rightarrow q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

> **Theorem I.** $[\ p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$
> **Theorem II.** $p \rightarrow p$

7. (20 pts) Prove the following theorem directly. (I.e. you cannot prove by the method of truth tables)

$$[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (q \vee r)]$$

8. (15 pts) Define the universe of elements $\mathbb{U}$ to be

$$\mathbb{U} = \left\{\, 2^k \;\middle|\; k = \ldots - 2, -1, 0, 1, 2, \ldots \right\}$$

Furthermore, define the operation of multiplication $\cdot$ as usual. Prove that $(\mathbb{U}, \cdot)$ satisfies all the requirements of being an Abelian group.

9. (30 pts) Define axiomatic systems (A) and (B) on a class K with relation R as follows:

Axiomatic system (A):
Axiom 1^A**.** The relation R is connected in the class K.
Axiom 2^A**.** The relation R is asymmetric in the class K.
Axiom 3^A**.** The relation R is transitive in the class K.

Axiomatic system (B):
Axiom 1^B**.** The relation R is connected in the class K.
Axiom 2^B**.** $(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \rightarrow \sim vRx$

Recall that Axiomatic system (A) implies that the relation R orders the class K. Prove that axiomatic systems (A) and (B) are equipollent. What does this imply about the definition of an ordering of a class K using a relation R?

10. (10 pts) Consider the following system of Axioms, where R is an arbitrary relation:

Axiom I. $\forall x \in \mathbf{S}\ (xRx)$
Axiom II. $\forall y, z \in \mathbf{S}\ (yRz \rightarrow zRy)$
Axiom III. $\forall x, y, z \in \mathbf{S}\ ((xRy \wedge yRz) \rightarrow xRz)$

Exhibit models of each of these three axioms such that:

(a) The first two sentences of the system, but not the last.
(b) The first and third sentence, but not the second.
(c) The last two sentences, but not the first.

Lastly, explain what this implies about the system of axioms.