Math 2143 - Brief Calculus with Applications

Exam #1 - 2017.09.29

Solutions

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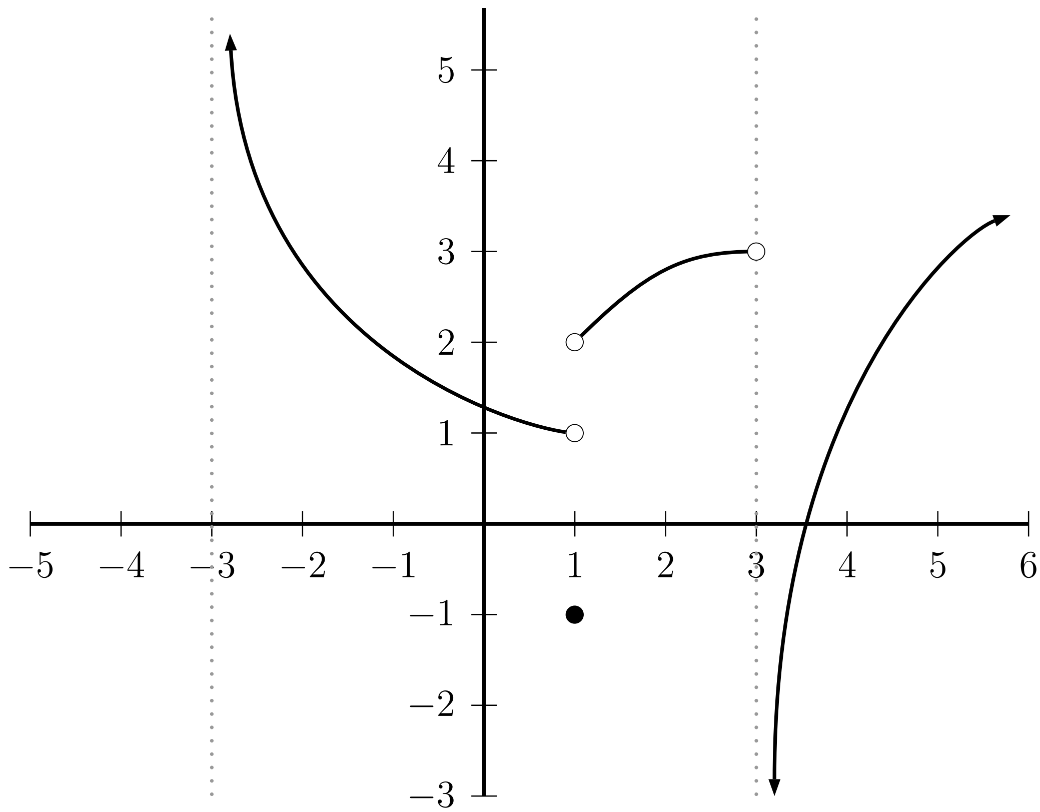
1. Sketch the graph of a single function $f(x)$ which satisfies the following properties:

   (a) Domain is $(-3,3)\cup(3,\infty)$
   (b) $\lim\limits_{x\to-3^+} f(x)=\infty$
   (c) $\lim\limits_{x\to1^-} f(x)=1$
   (d) $\lim\limits_{x\to1^+} f(x)=2$
   (e) $f(1)=-1$
   (f) $\lim\limits_{x\to3^-} f(x)=3$
   (g) $\lim\limits_{x\to3^+} f(x)=-\infty$

Answers will vary, here is one example...



2. (10 pts. each) Compute the following limits. If the limit does not exist, but is ∞ or $-\infty$, state which one.

(a) $\displaystyle \lim_{x\to 2} \frac{2x^2-3x-2}{x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{2x^2-3x-2}{x-2} &= \lim_{x\to 2} \frac{(x-2)(2x+1)}{x-2} \\
&= \lim_{x\to 2} (2x+1)\cdot \frac{x-2}{x-2} \\
&= \lim_{x\to 2} (2x+1)\cdot \lim_{x\to 2} \frac{x-2}{x-2} \\
&= 5\cdot 1 \\
&= 5
\end{aligned}$$

(b) $\displaystyle \lim_{h\to 0} \frac{\frac{1}{2h+3}-\frac{1}{3}}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{2h+3}-\frac{1}{3}}{h} &= \lim_{h\to 0} \frac{\frac{3-(2h+3)}{3(2h+3)}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{3h(2h+3)} \\
&= \lim_{h\to 0} \frac{-2}{3(2h+3)} \\
&= -\frac{2}{9}
\end{aligned}$$

(c) $\displaystyle \lim_{h\to 0} \frac{\sqrt{2h+1}-1}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{2h+1}-1}{h} &= \lim_{h\to 0} \frac{\sqrt{2h+1}-1}{h}\cdot \frac{\sqrt{2h+1}+1}{\sqrt{2h+1}+1} \\
&= \lim_{h\to 0} \frac{(2h+1)-1}{h(\sqrt{2h+1}+1)} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2h+1}+1)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2h+1}+1} \\
&= 1
\end{aligned}$$

(d) $\displaystyle \lim_{x\to 3^+} \frac{(x+2)(x+1)}{(x-3)(x-2)}$

Note that if we plug in $x=3$, we end up with $\frac{20}{0}$, so the limit is going to be either ∞ or $-\infty$. Plugging in numbers close to $x=3$ but greater than 3 gives something close to 20 in the numerator, and in the denominator, something close to 0 but positive times something close 1, thus we have

$$\lim_{x\to 2^-} \frac{(x+2)(x+1)}{(x-3)(x-2)} = \frac{(\approx 5)\cdot(\approx 4)}{\text{small positive number}\cdot(\approx 1)} = +\infty$$

3. Compute the following derivatives (do not simplify your answer):

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2-4\sqrt[3]{x}+3}{x^{4/3}+2x-\sqrt[6]{x^5}}$

We apply the quotient rule here, $f(x)=x^2-4\sqrt[3]{x}+3$, $g(x)=x^{4/3}+2x-\sqrt[6]{x^5}$, and $f'(x)=2x-\frac{4}{3}x^{-2/3}$ and $g'(x)=\frac{4}{3}x^{1/3}+2-\frac{5}{6}x^{-1/6}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2-4x+3}{x^2+2x-1}=\frac{\left(2x-\frac{4}{3}x^{-2/3}\right)\left(x^{4/3}+2x-\sqrt[6]{x^5}\right)-\left(x^2-4\sqrt[3]{x}+3\right)\left(\frac{4}{3}x^{1/3}+2-\frac{5}{6}x^{-1/6}\right)}{\left(x^{4/3}+2x-\sqrt[6]{x^5}\right)^2}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt[5]{\left(3\sqrt{x}-2x+1\right)\left(5+\sqrt[7]{x}-x^3\right)}$

We apply the chain rule, with $f(x)=\sqrt[5]{x}$ and $g(x)=\left(3\sqrt{x}-2x+1\right)\left(5+\sqrt[7]{x}-x^3\right)$. This gives $f'(x)=\frac{1}{5}x^{-4/5}$, and we have to use the product rule on $g(x)$:

$$\begin{aligned}g'(x)&=\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(3\sqrt{x}-2x+1\right)\right]\left(5+\sqrt[7]{x}-x^3\right)+\left(3\sqrt{x}-2x+1\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(5+\sqrt[7]{x}-x^3\right)\right]\\&=\left(\frac{3}{2}x^{-1/2}-2\right)\left(5+\sqrt[7]{x}-x^3\right)+\left(3\sqrt{x}-2x+1\right)\left(\frac{1}{7}x^{-6/7}-3x^2\right)\end{aligned}$$

So now we put it all together:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[5]{\left(3\sqrt{x}-2x+1\right)\left(5+\sqrt[7]{x}-x^3\right)}&=\frac{1}{5}\left(\left(3\sqrt{x}-2x+1\right)\left(5+\sqrt[7]{x}-x^3\right)\right)^{-4/5}\cdot\\&\quad\left(\frac{3}{2}x^{-1/2}-2\right)\left(5+\sqrt[7]{x}-x^3\right)+\left(3\sqrt{x}-2x+1\right)\left(\frac{1}{7}x^{-6/7}-3x^2\right)\end{aligned}$$

(c) $\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}\left(x^2-\dfrac{3}{x}+\dfrac{2}{x^2}\right)$

We need to take three derivatives, so we start with the first:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-\frac{3}{x}+\frac{2}{x^2}\right)=2x+\frac{3}{x^2}-\frac{4}{x^3}$$

Next we take another derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2x+\frac{3}{x^2}-\frac{4}{x^3}\right)=2-\frac{6}{x^3}+\frac{12}{x^4}$$

One final derivative gives

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(x^2-\frac{3}{x}+\frac{2}{x^2}\right)=\frac{18}{x^4}-\frac{48}{x^5}$$

4. Find the equation of the tangent line to $f(x)=x^3-3\sqrt{x}+\dfrac{4}{x^2}-2$ at $x=1$.

We use the point-slope form of a line: $y-y_0=m(x-x_0)$, with $y_0=f(x_0)$ and $m=f'(x_0)$. First, we find $f(x_0)=f(1)=0$, and then we compute $f'(x)$:

$$f'(x)=3x^2-\frac{3}{2\sqrt{x}}-\frac{8}{x^3}$$

Next, using $f'(x)$ compute above, we find $f'(x_0)=f'(1)=-13/2$. So we get the equation of the tangent line is:

$$y=-\frac{13}{2}(x-1)$$