Math 2215 - Calculus 1

Exam #1 - 2017.09.07

Solutions

For problems 1–11, compute each limit. If the limit does not exist, differentiate between $+\infty$, $-\infty$, or does not exist.

1. $\lim\limits_{x\to\infty} \sin\left(\dfrac{x}{1+x^2}\right)$

$$\begin{aligned}\lim_{x\to\infty} \sin\left(\frac{x}{1+x^2}\right) &= \sin\left(\lim_{x\to\infty} \frac{x}{1+x^2}\right)\\ &= \sin(0)\\ &= 0\end{aligned}$$

2. $\lim\limits_{x\to 0} \sin\left(\dfrac{x}{1+x^2}\right)$

$$\begin{aligned}\lim_{x\to 0} \sin\left(\frac{x}{1+x^2}\right) &= \sin\left(\lim_{x\to 0} \frac{x}{1+x^2}\right)\\ &= \sin(0)\\ &= 0\end{aligned}$$

3. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2x-1}}{3x-7}$

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2+2x-1}}{3x-7} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{\sqrt{x^2+2x-1}}{3x-7}\\ &= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}\left(x^2+2x-1\right)}}{3-\frac{7}{x}}\\ &= \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}}{3-\frac{7}{x}}\\ &= \frac{1}{3}\end{aligned}$$

4. $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2+2x-1}}{3x-7}$

$$\begin{aligned}\lim_{x\to -\infty} \frac{\sqrt{x^2+2x-1}}{3x-7} &= \lim_{x\to -\infty} \frac{-\frac{1}{|x|}}{\frac{1}{x}} \cdot \frac{\sqrt{x^2+2x-1}}{3x-7}\\ &= \lim_{x\to -\infty} \frac{-\sqrt{\frac{1}{x^2}\left(x^2+2x-1\right)}}{3-\frac{7}{x}}\\ &= \lim_{x\to -\infty} \frac{-\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}}{3-\frac{7}{x}}\\ &= -\frac{1}{3}\end{aligned}$$

5. $\lim\limits_{x\to 0} \dfrac{x^2}{\cos(x)}$

$$\lim_{x\to 0} \frac{x^2}{\cos(x)} = 0$$

6. $\lim_{x\to 0} \frac{\cos(x)}{x^2}$

This limit does not exist, since it is of the form $\frac{1}{0}$, but it does approach $+\infty$ from both sides since $\cos(x) \to 1$ as $x \to 0$, and $x^2 > 0$.

7. $\lim_{x\to 0} \frac{\sin(3x)}{4x}$

We know that

$$\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1,$$

so if we multiply the original limit by 1 in the form $\frac{3}{3}$:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x)}{4x} &= \lim_{x\to 0} \frac{3}{3}\frac{\sin(3x)}{4x} \\
&= \lim_{x\to 0} \frac{3}{4}\frac{\sin(3x)}{3x} \\
&= \frac{3}{4}\lim_{x\to 0} \frac{\sin(3x)}{3x} \\
&= \frac{3}{4} \cdot 1 \\
&= \frac{3}{4}
\end{aligned}$$

8. $\lim_{x\to 2^+} \frac{(2x-3)(x-3)}{(x-2)(x+3)}$

Here we end up with the form $\frac{\text{nonzero}}{0}$. Thus, the limit will be $+\infty$ or $-\infty$. Plugging in values close to 2 but greater, we have $\frac{+-}{++}$ which is negative, so

$$\lim_{x\to 2^+} \frac{(2x-3)(x-3)}{(x-2)(x+3)} = -\infty$$

9. $\lim_{x\to 2^-} \frac{(2x-3)(x-3)}{(x-2)(x+3)}$

Here we end up with the form $\frac{\text{nonzero}}{0}$. Thus, the limit will be $+\infty$ or $-\infty$. Plugging in values close to 2 but less than 2, we have $\frac{+-}{-+}$ which is positive, so

$$\lim_{x\to 2^-} \frac{(2x-3)(x-3)}{(x-2)(x+3)} = +\infty$$

10. $\lim_{x\to 3^-} \frac{(2x-3)(x-3)}{(x-2)(x+3)}$

Since $x = 3$ is in the domain of this rational function, we can just plug it in!

$$\lim_{x\to 3^-} \frac{(2x-3)(x-3)}{(x-2)(x+3)} = 0.$$

11. $\lim_{x\to \infty} \frac{(2x-3)(x-3)}{(x-2)(x+3)}$

If we multiply out the numerator and denominator, the leading term in each is $2x^2$ and x^2, respectively. Since the degrees are equal, the infinite limit is simply the ratio of the two leading coefficients.

$$\lim_{x\to \infty} \frac{(2x-3)(x-3)}{(x-2)(x+3)} = \frac{2}{1} = 2.$$

12. Find values of a and b so that the following piecewise function is continuous everywhere.

$$f(x) = \begin{cases} \frac{2}{\pi}x - a, & x \leq -\pi \\ b\cos(x) + 4, & -\pi < x < \pi \\ 3\sin\left(\frac{x}{2}\right) - 1, & x \geq \pi \end{cases}$$

Since each piece is continuous on its domain, we simply need to ensure continuity at $x = -\pi$ and $x = \pi$. Let's first look at $x = \pi$, where $f(\pi) = 3\sin\left(\frac{\pi}{2}\right) - 1 = 2$. We also know from computing $f(\pi)$ that

$$\lim_{x \to \pi^+} f(x) = 2.$$

Now, we need to compute the limit from the left, which we use the middle function, $b\cos(x) + 4$:

$$\begin{aligned} \lim_{x \to \pi^-} f(x) &= \lim_{x \to \pi^-} b\cos(x) + 4 \\ &= -b + 4. \end{aligned}$$

Setting $-b + 4 = 2$ gives $b = 2$.

For $x = -\pi$, we have $f(-\pi) = \frac{2}{\pi}(-\pi) - a = -2 - a$. Similarly, computing the limit as $x \to -\pi^-$ uses the same piece we just evaluated, so

$$\lim_{x \to -\pi^-} f(x) = -2 - a.$$

Lastly, we compute the limit from the right:

$$\begin{aligned} \lim_{x \to -\pi^+} f(x) &= \lim_{x \to -\pi^+} 2\cos(x) + 4 \\ &= -2 + 4 \\ &= 2. \end{aligned}$$

Setting $-2 - a = 2$ gives $a = -4$. Our piecewise function $f(x)$ will now be continuous on the entire real line.

$$f(x) = \begin{cases} \frac{2}{\pi}x + 4, & x \leq -\pi \\ 2\cos(x) + 4, & -\pi < x < \pi \\ 3\sin\left(\frac{x}{2}\right) - 1, & x \geq \pi \end{cases}$$

13. $\tan(\pi/4) = 1$

14. $\tan^2(x) + 1 = \sec^2(x)$