Math 2143 - Brief Calculus with Applications

Exam #2 - 2017.10.25

Solutions

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1. If $f(x) = \frac{x^6}{6} + \frac{2x^5}{5} - \frac{x^4}{4} - \frac{2x^3}{3} - 7$, compute $f'(x)$, find critical points, and classify them as local max/mins or neither after determining the intervals of increase and decrease.

First we take the derivative:

$$\begin{aligned} f'(x) &= x^5 + 2x^4 - x^3 - 2x^2 \\ &= x^2(x-1)(x+1)(x+2) \end{aligned}$$

So critical points are at $x = -2$, $x = -1$, $x = 0$ and $x = 1$.

To find the intervals of increase and decrease note that if x is large, $f'(x) >$, with a change in sign occurring at $x = 1$, but not at $x = 0$, and again changes in sign at $x = -1$ and $x = -2$. So:

$$\begin{aligned} &f(x) \text{ is increasing on } (-2,-1) \cup (1,\infty), \\ &f(x) \text{ is decreasing on } (-\infty,-2) \cup (-1,0) \cup (0,1). \end{aligned}$$

Using the intervals of increase and decrease, we see that there are local mins at $x = -2$ and $x = 1$, and a local max at $x = -1$. The critical point at $x = 0$ is neither a local max nor a local min.

2. For $g(x) = \frac{x}{x^2+1}$, it can be shown that (you do not have to) the first and second derivatives are:

$$g'(x) = \frac{1-x^2}{(1+x^2)^2}, \quad g''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}$$

Use $g'(x)$ to locate critical points and $g''(x)$ to find intervals of concavity. Furthermore, use concavity to determine if the critical points are local max or mins.

Critical points are were the $g'(x) = 0$ or does not exist. Setting the numerator of $g'(x) = 0$ gives $x = \pm 1$. The denominator is never zero, thus there are no more critical points than the two already located.

To find intervals of concavity, we first find all the values where $g''(x) = 0$ or does not exist. Similar to $g'(x)$, the denominator of $g''(x)$ is never zero. So we simply set the numerator to zero, which gives $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$. Since each of the roots have multiplicity 1, we know that concavity will change sign at each of the three points. Plugging a large positive x into $g''(x)$ yields a positive result. So

$$\begin{aligned} &g(x) \text{ is concave up on } (-\sqrt{3},0) \cup (\sqrt{3},\infty), \\ &g(x) \text{ is concave down } (-\infty,-\sqrt{3}) \cup (0,\sqrt{3}). \end{aligned}$$

Since $x = -1$ is on the interval $(-\sqrt{3},0)$ which is concave up, $x = -1$ is a local min. Similarly, $x = 1$ is on the interval $(0,\sqrt{3})$ which is concave down, so $x = 1$ is a local max.

3. For this problem, let $h(x) = \frac{(x-2)^2}{x^2+4}$.

(a) Compute the domain of $h(x)$.

The domain of $h(x)$ is all real numbers. There are no zeros in the denominator, and $x^2 + 4$ is always positive.

(b) Determine any horizontal or vertical asymptotes.

There are no vertical asymptotes, so all we need is a horizontal asymptote. If we look at the limit as $x \to \pm\infty$, we simply compare leading powers in the numerator and denominator. The degree of the numerator is 2, and in the denominator is 2, thus there is a horizontal asymptote of $y = 1$ which is the ratio of the leading coefficients.

(c) Compute the roots and the y-intercept.

The only root is at $x = 2$, which is the point $(2, 0)$. The y-intercept as at $(0, 1)$.

(d) Compute $h'(x)$, and verify that $h'(x) = \dfrac{4(x^2-4)}{(x^2+4)^2}$.

We use the quotient rule here:

$$\begin{aligned} h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{4(x^2-4)}{(x^2+4)^2} &= \frac{2(x-2)\cdot(x^2+4)-(x-2)^2\cdot 2x}{(x^2+4)^2} \\ &= \frac{2(x-2)(x^2+4-x(x-2))}{(x^2+4)^2} \\ &= \frac{2(x-2)(4+2x)}{(x^2+4)^2} \\ &= \frac{4(x-2)(x+2)}{(x^2+4)^2} \\ &= \frac{4(x^2-4)}{(x^2+4)^2} \end{aligned}$$

(e) Locate all critical points of $h(x)$.

Since the denominator is still never zero, critical points are only where the numerator is zero, so $x = \pm 2$ are the only critical points.

(f) Determine the intervals of increase and decrease of $h(x)$.

Since the numerator solely determines the sign of the derivative, and the numerator is a parabola opening up with roots at $x = -2$ and $x = 2$, we have that $h(x)$ is decreasing on $(-2, 2)$ and increasing on $(-\infty, -2) \cup (2, \infty)$.

(g) Classify all the critical points from (e) using your answer to part (f).

Since $h(x)$ goes from increasing to decreasing at $x = -2$, $x = -2$ is a local maximum. By a similar argument (but opposite), $x = 2$ is a local minimum.

(h) Compute $h''(x)$, and verify that $h''(x) = \dfrac{-8x(x^2-12)}{(x^2+4)^3}$.

We use the quotient rule again:

$$\begin{aligned} h''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{-8x(x^2-12)}{(x^2+4)^3} &= 4\cdot\frac{2x\cdot(x^2+4)^2-(x^2-4)\cdot 2(x^2+4)\cdot 2x}{(x^2+4)^4} \\ &= 4\cdot\frac{2x(x^2+4)-4x(x^2-4)}{(x^2+4)^3} \\ &= \frac{8x(x^2+4-2(x^2-4))}{(x^2+4)^3} \\ &= \frac{8x(-x^2+12)}{(x^2+4)^3} \\ &= \frac{-8x(x^2-12)}{(x^2+4)^3} \end{aligned}$$

(i) Determine the intervals of concavity for $h(x)$.

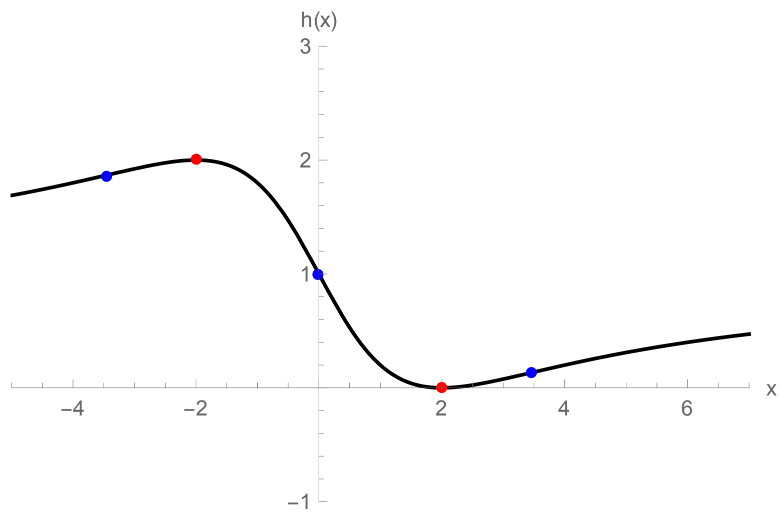
Once again, the denominator is always positive, so we consider only the numerator. The numerator is zero at $x = 0$ and $x = \pm\sqrt{12} = \pm 2\sqrt{3}$, and can be thought of as the product of a line passing through the origin with negative slope ($y = -8x$) and a parabola opening upwards with roots at $x = \pm 2\sqrt{3}$ ($y = x^2 - 12$). Or, one could test the points $x = -4, -1, 1, 4$. We have that $h(x)$ is concave up on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$, and concave down on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$.

(j) Using your answer to part (i), state the inflection points of $h(x)$.

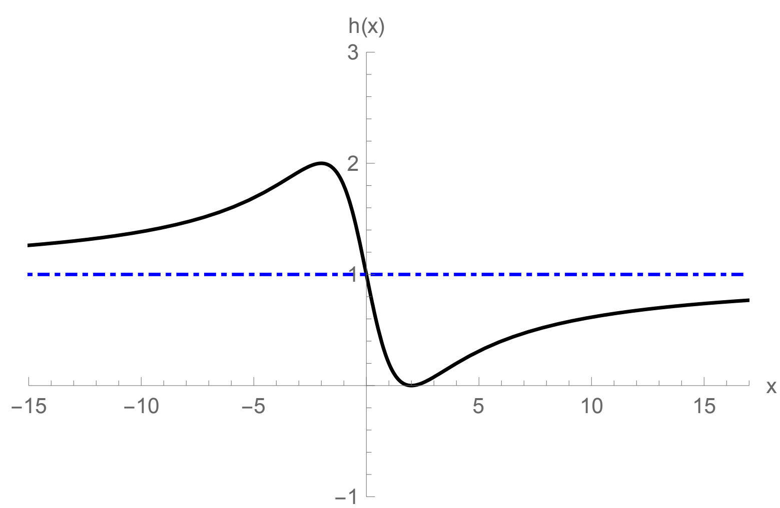
There is a change in inflection at $x = 0$, and $x = \pm 2\sqrt{3}$.

(k) Use all the information from parts (a)–(j) to sketch a graph of $h(x)$.

In the following figure, the blue dots correspond to inflection points, and the red dots to critical points.



The next graph shows more of the limiting behaviour as x gets further away from the origin. The blue dot-dashed line is the horizontal asymptote of $y = 1$.



4. Find the equation of the tangent line to the implicitly defined equation $xy + y^2 - 2x = 0$ at the point $(x, y) = (1, -2)$ by using implicit differentiation.

By treating $y = f(x)$ we differentiate both sides with respect to x and solve for $f'(x) = y'$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(xy + y^2 - 2x) &= \frac{\mathrm{d}}{\mathrm{d}x}0 \\
1 \cdot y + x \cdot y' + 2yy' - 2 &= 0 \\
y'(x + 2y) &= 2 - y \\
y' &= \frac{2 - y}{x + 2y}.
\end{aligned}$$

Now we plug in $(x, y) = (1, -2)$ into y' to get $y'(1, -2) = -\frac{4}{3}$. So the point-slope form of the equation of the tangent line should be given by

$$y + 2 = -\frac{4}{3}(x - 1).$$