Math 2215 - Calculus 1

Exam #2 - 2017.09.28

Solutions

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1. Fill out the following table of derivatives for the given six trigonometric functions.

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$

2. Fill out the following table of values for the given trigonometric functions at the given points:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1

3. State the Mean Value Theorem.

If $f(x)$ is defined and continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For problems 4–7, compute each derivative.

4. $\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sec(\theta^2)\ \sqrt{\theta^2+1}\right]$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sec(\theta^2)\ \sqrt{\theta^2+1}\right] &= \left[\frac{\mathrm{d}}{\mathrm{d}\theta}\sec^2(\theta)\right]\cdot\sqrt{\theta^2+1} + sec^2(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\sqrt{\theta^2+1} \\ &= \sec(\theta^2)\tan(\theta^2)\cdot 2\theta\cdot\sqrt{\theta^2+1} + \sec(\theta^2)\cdot\frac{2\theta}{2\sqrt{\theta^2+1}}\end{aligned}$$

5. $\frac{\mathrm{d}}{\mathrm{d}z}\left(1 + \left(\frac{z}{2\sin(z)}\right)^4\right)^3$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left(1+\left(\frac{z}{2\sin(z)}\right)^4\right)^3 &= 3\left(1+\left(\frac{z}{2\sin(z)}\right)^4\right)^2 \cdot \frac{\mathrm{d}}{\mathrm{d}z}\left[1+\left(\frac{z}{2\sin(z)}\right)^4\right] \\
&= 3\left(1+\left(\frac{z}{2\sin(z)}\right)^4\right)^2 \cdot \left[4\left(\frac{z}{2\sin(z)}\right)^3 \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{2\sin(z)}\right)\right] \\
&= 3\left(1+\left(\frac{z}{2\sin(z)}\right)^4\right)^2 \cdot \left[4\left(\frac{z}{2\sin(z)}\right)^3 \left(\frac{2\sin(z)-2z\cos(z)}{4\sin^2(z)}\right)\right]
\end{aligned}$$

6. $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[t^2\sin^4(t)\cos^5(2t)\right]$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[t^2\sin^4(t)\cos^5(2t)\right] &= \left[\frac{\mathrm{d}}{\mathrm{d}t}t^2\right]\sin^4(t)\cos^5(2t) + t^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\sin^4(t)\right]\cos^5(2t) + t^2\sin^4(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}\cos^5(2t)\right] \\
&= 2t\sin^4(t)\cos^5(2t) + t^2 4\sin^3(t)\cos(t)\cos^5(2t) \\
&\qquad + t^2\sin^4(t)\left[5\cos^4(2t)\cdot(-\sin(2t)\cdot 2)\right]
\end{aligned}$$

7. $\dfrac{\mathrm{d}}{\mathrm{d}w}\, f(g(h(j^2(w){\cdot}k(w))))$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}w}\, f(g(h(j^2(w)\cdot k(w)))) \\
&\quad = f'(g(h(j^2(w)\cdot k(w))))\cdot \frac{\mathrm{d}}{\mathrm{d}w} g(h(j^2(w)\cdot k(w))) \\
&\quad = f'(g(h(j^2(w)\cdot k(w))))\cdot g'(h(j^2(w)\cdot k(w)))\cdot \frac{\mathrm{d}}{\mathrm{d}w} h(j^2(w)\cdot k(w)) \\
&\quad = f'(g(h(j^2(w)\cdot k(w))))\cdot g'(h(j^2(w)\cdot k(w)))\cdot h'(j^2(w)\cdot k(w))\cdot \frac{\mathrm{d}}{\mathrm{d}w}\left[j^2(w)\cdot k(w)\right] \\
&\quad = f'(g(h(j^2(w)\cdot k(w))))\cdot g'(h(j^2(w)\cdot k(w)))\cdot h'(j^2(w)\cdot k(w))\cdot \left[2j(w)\cdot j'(w)\cdot k(w) + j^2(w)\cdot k'(w)\right]
\end{aligned}$$

For problems 8–10, use the implicitly defined equation $\cos(x)+\cos(2y) = \dfrac{1}{2}$.

8. Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

   First we take a derivative with respect to x on both sides.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)+\cos(2y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2} \\
-\sin(x) - 2\sin(2y)y' &= 0
\end{aligned}$$

Solving for y' gives

$$y' = -\frac{\sin(x)}{2\sin(2y)}$$

9. Compute the equation of the tangent line to the implicitly defined function at $(x,y) = (\pi/3, -\pi/4)$.

   We use the point-slope equation of the tangent line: $y - y_0 = m(x - x_0)$, where $x_0 = \pi/3$, $y_0 = -\pi/4$ and

$$m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(\pi/3,-\pi/4)} = -\frac{\sin(\pi/3)}{2\sin(-\pi/2)} = \frac{\sqrt{3}}{4}$$

So the equation of the tangent line is

$$y + \frac{\pi}{4} = \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)$$

10. Find *all* points (x, y) on the curve where there is a vertical tangent line.

To find where there are vertical tangent lines, we simply set the denominator to zero and make sure that the numerator is not zero at the same time. So solving $2\sin(2y) = 0$ gives

$$y = \frac{\pi}{2} \cdot k, k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Now we need to find the corresponding $x-$coordinates that go with each value of y. Using our original implicitly defined function, we have

$$\cos(x) + \cos\left(2 \cdot \frac{\pi}{2}k\right) = \frac{1}{2}$$
$$\cos(x) + \cos(\pi k) = \frac{1}{2}$$

Now notice that $\cos(\pi k) = (-1)^k$, so

$$\cos(x) = \frac{1}{2} - (-1)^k$$

Note that if k is even, we end up with $\cos(x) = -\frac{1}{2}$. However, if k is odd, we have $\cos(x) = \frac{3}{2}$, which has no solution. So now we have reduced our possible $y-$ values to those which are just multiples of π. And solving $\cos(x) = -\frac{1}{2}$ gives $x = 2\pi/3 + 2\pi k$ and $x = 4\pi/3 + 2\pi k$. Thus the points are of the form:

$$(x, y) = (2\pi/3 + 2\pi k, \pi l), \quad (x, y) = (4\pi/3 + 2\pi k, \pi l)$$

Where k and l are any integers.

The following figure shows the graph of $\cos(x) + \cos(2y) = \dfrac{1}{2}$ in blue, with the tangent line at $(x, y) = (\pi/3, -\pi/4)$ in black, and the points with a vertical tangent line given by the formulas for (x, y) in the answer to problem 10 are in red.