Math 2215 - Calculus 1

Exam #3 - 2017.10.23

Solutions

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1. Approximate $\sqrt[3]{0.99}$ and $\sqrt[3]{1.04}$ using the linearization of $f(x) = \sqrt[3]{1+x}$.

   We can compute the linearization of $f(x)$ at $x = 0$ and use it to evaluate the two cube roots in question. First

$$f'(x) = \frac{1}{3}(1+x)^{-2/3},$$

and thus $f'(0) = \dfrac{1}{3}$. The linearizaton $L(x)$ at $x = 0$ is thus given by

$$L(x) = \frac{1}{3}(x-1) + 1 = \frac{1}{3}x + 1$$

   To approximate $\sqrt[3]{0.99}$, we set $x = -0.01$ in $L(x)$:

$$\begin{aligned} L(-0.01) &= \frac{1}{3}(-0.01) + 1 \\ &= 1 - \frac{1}{300} \\ &= \frac{299}{300} \end{aligned}$$

   To approximate $\sqrt[3]{1.04}$, we set $x = 0.04$ in $L(x)$:

$$\begin{aligned} L(-0.01) &= \frac{1}{3}(0.04) + 1 \\ &= 1 + \frac{4}{300} \\ &= \frac{304}{300} \end{aligned}$$

   Both of the approximations are overestimates, since the graph of $f(x) = \sqrt[3]{1+x}$ lies below the tangent line (i.e. $f(x)$ is concave down).

2. Determine the global maximum and global minimum of $g(x) = x\sqrt{1+x}$ on its domain.

   First, the domain of $g(x)$ is $[-1, \infty)$, with

$$g(-1) = 0 \quad \text{and} \quad \lim_{x\to\infty} g(x) = +\infty$$

Since the limit above is $+\infty$, there will be no global maximum, only a global minimum. So we now compute the derivative:

$$\begin{aligned} g'(x) &= 1 \cdot \sqrt{1+x} + \frac{x}{2\sqrt{1+x}} \\ &= \frac{2+3x}{2\sqrt{1+x}} \end{aligned}$$

The only critical points are at $x = -1$ and $x = -\dfrac{2}{3}$. We know the function value at $x = -1$ already, so $g(-2/3) = -\dfrac{2}{3\sqrt{3}}$. Clearly the global minimum occurs at $x = -2/3$, with a value of $-\dfrac{2}{3\sqrt{3}}$.

3. For this problem, let $h(x) = \dfrac{x}{(1+3x^2)^{2/3}}$.

   (a) Compute the domain of $h(x)$.

The domain of $h(x)$ is all real numbers. There are no zeros in the denominator, and $1 + 3x^2$ is always positive.

(b) Determine any horizontal or vertical asymptotes.

There are no vertical asymptotes, so all we need is a horizontal asymptote. If we look at the limit as $x \to \pm\infty$, we simply compare leading powers in the numerator and denominator. The degree of the numerator is 1, and in the denominator is 4/3, thus there is a horizontal asymptote of $y = 0$.

(c) Compute the roots and the y-intercept.

The only root (and y-intercept) as at the origin $(0, 0)$.

(d) Compute $h'(x)$, and verify that $h'(x) = \dfrac{1 - x^2}{(1 + 3x^2)^{5/3}}$.

We use the quotient rule here:

$$
\begin{aligned}
h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{(1+3x^2)^{2/3}} &= \frac{1 \cdot (1+3x^2)^{2/3} - x \cdot \frac{2}{3}(1+3x^2)^{-1/3} \cdot 6x}{(1+3x^2)^{4/3}} \\
&= \frac{(1+3x^2)^{2/3} - 6x^2 \cdot \frac{2}{3}(1+3x^2)^{-1/3}}{(1+3x^2)^{4/3}} \cdot \frac{(1+3x^2)^{1/3}}{(1+3x^2)^{1/3}} \\
&= \frac{(1+3x^2) - 6x^2 \cdot \frac{2}{3}}{(1+3x^2)^{5/3}} \\
&= \frac{1-x^2}{(1+3x^2)^{5/3}}
\end{aligned}
$$

(e) Locate all critical points of $h(x)$.

Since the denominator is still never zero, critical points are only where the numerator is zero, so $x = \pm 1$ are the only critical points.

(f) Determine the intervals of increase and decrease of $h(x)$.

Since the numerator solely determines the sign of the derivative, and the numerator is a parabola opening down with roots at $x = -1$ and $x = 1$, we have that $h(x)$ is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.

(g) Classify all the critical points from (e) using your answer to part (f).

Since $h(x)$ goes from decreasing to increasing at $x = -1$, $x = -1$ is a local minimum. By a similar argument (but opposite), $x = 1$ is a local maximum.

(h) Compute $h''(x)$, and verify that $h''(x) = \dfrac{4x(x^2 - 3)}{(1 + 3x^2)^{8/3}}$.

We use the quotient rule again:

$$
\begin{aligned}
h''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1-x^2}{(1+3x^2)^{5/3}} &= \frac{-2x\cdot(1+3x^2)^{5/3} - (1-x^2)\cdot\frac{5}{3}(1+3x^2)^{2/3}\cdot 6x}{(1+3x^2)^{10/3}} \\
&= \frac{-2x\cdot(1+3x^2) - (1-x^2)\cdot\frac{5}{3}\cdot 6x}{(1+3x^2)^{8/3}} \\
&= \frac{-2x\cdot(1+3x^2) - 10x\cdot(1-x^2)}{(1+3x^2)^{8/3}} \\
&= \frac{-12x+4x^2}{(1+3x^2)^{8/3}} \\
&= \frac{4x(x^2-3)}{(1+3x^2)^{8/3}}
\end{aligned}
$$

(i) Determine the intervals of concavity for $h(x)$.

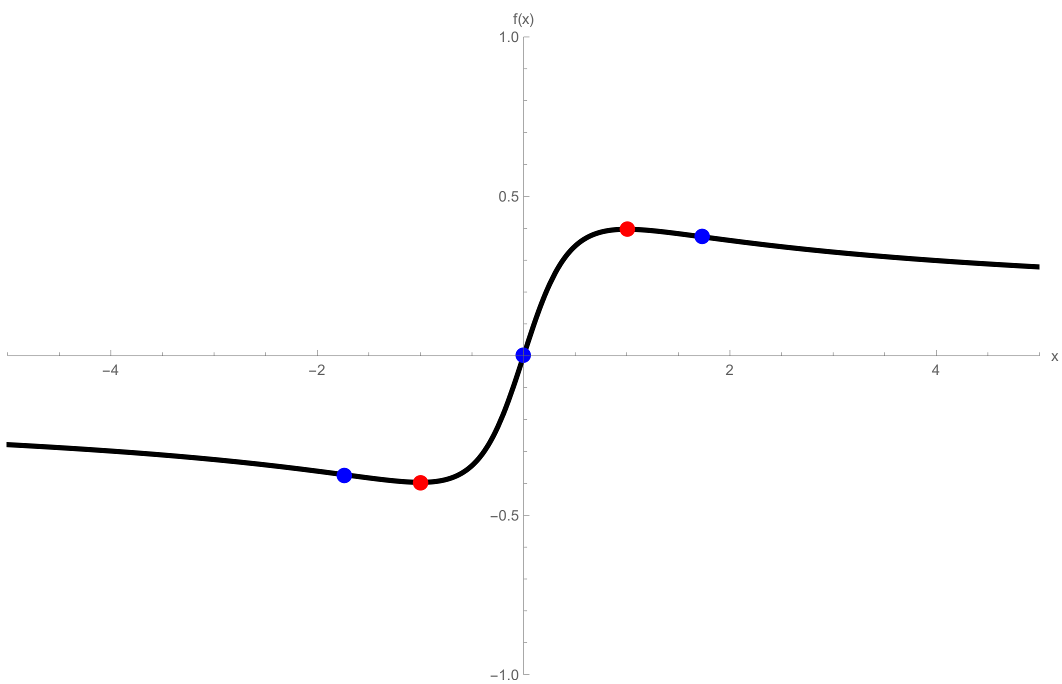
Once again, the denominator is always positive, so we consider only the numerator. The numerator is zero at $x = 0$ and $x = \pm\sqrt{3}$, and can be thought of as the product of a line passing through the origin with positive slope ($y = 4x$) and a parabola opening upwards with roots at $x = \pm\sqrt{3}$ ($y = x^2 - 3$). Or, one could test the points $x = -2, -1, 1, 2$. We have that $h(x)$ is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$, and concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.

(j) Using your answer to part (i), state the inflection points of $h(x)$.

There is a change in inflection at $x = 0$, and $x = \pm\sqrt{3}$.

(k) Use all the information from parts (a)–(j) to sketch a graph of $h(x)$.

In the following figure, the blue dots correspond to inflection points, and the red dots to critical points.



4. Corn syrup is pouring onto your floor at a rate of 1 in^3/sec into a completely circular puddle which is uniformly 1/2″ thick. The closest wall is 3 feet away. At what rate is the radius of the corn syrup spill growing when it finally

reaches the wall?

First we need to determine the formula for the volume of the spill, which is a cylinder (even if it is flat). This is $V = \pi r^2 h$. In this case, V and r are both change over time, while h does not. Thus

$$V(t) = \pi\ h\ r^2(t)$$

Since we want $r'(t)$ when the puddle hits the wall, we compute a time derivative of the above equation:

$$V'(t) = 2\pi\ h\ r(t) r'(t)$$

We know that $V'(t) = 1$ in^3/sec, and when the puddle hits the wall, $r(t) = 36$ in and $h = 1/2$ in. Solving for $r'(t)$ gives

$$\begin{aligned} r'(t) &= \frac{V'(t)}{2\pi\ h\ r(t)} \\ &= \frac{1 \text{ in}^3/\text{sec}}{2\pi\ 1/2 \text{ in} \cdot 36 \text{ in}} \\ &= \frac{1}{36\pi}\ \frac{\text{in}}{\text{sec}} \end{aligned}$$