Math 2215 - Calculus 1

Exam #5 - 2017.12.04

Solutions

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1. Let $P(x) = -\dfrac{1}{4}(x+2)(x-3)$, and $S(x) = \sqrt{x-1}$. Find the area of the region bounded by $P(x)$, $S(x)$, and the x-axis which contains the point $(0,1)$. *Hint: Evaluate* $P(2)$ *and* $S(2)$.

The bounded region $\mathcal{R}$ between $P(x)$ and the x-axis is divided into two pieces, $\mathcal{R}_1$ and $\mathcal{R}_2$, by $S(x)$, as seen in Figure 1 below:

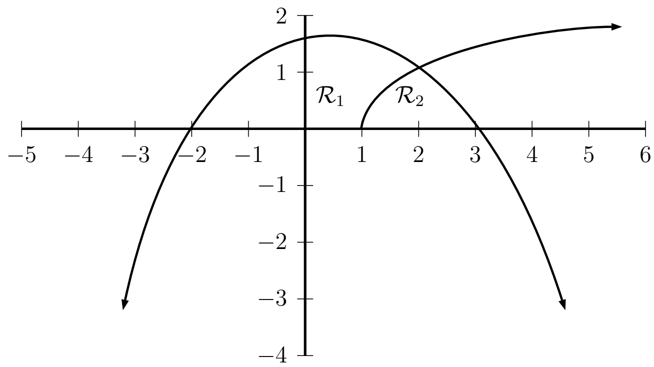

Figure 1. Graphs of $P(x)$, $S(x)$, and the two bounded regions $\mathcal{R}_1$ and $\mathcal{R}_2$ bounded by the x-axis.

Since $P(x)$ is already factored, we know it has roots at $x = -2$ and $x = 3$, while $S(x)$ is a square root function, opening to the right whose root is at $x = 1$. As per the hint, these two functions intersect at $x = 2$ and the region whose area needs to be computed is $\mathcal{R}_1$ as seen above in Figure 1. To compute this area, we will have to use two integrals:

$$
\begin{aligned}
\mathcal{A} &= \int_{-2}^{1} P(x) - 0\,\mathrm{d}x + \int_{1}^{2} P(x) - S(x)\,\mathrm{d}x \\
&= -\frac{1}{4}\int_{-2}^{1} x^2 - x - 6\,\mathrm{d}x + \int_{1}^{2} -\frac{1}{4}\left(x^2 - x - 6\right) - \sqrt{x-1}\,\mathrm{d}x \\
&= -\frac{1}{4}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]\Bigg|_{-2}^{1} + -\frac{1}{4}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]\Bigg|_{1}^{2} - \frac{2}{3}(x-1)^{3/2}\Bigg|_{1}^{2} \\
&= -\frac{1}{4}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]\Bigg|_{-2}^{2} - \frac{2}{3}(x-1)^{3/2}\Bigg|_{1}^{2} \\
&= \frac{14}{3} - \frac{2}{3} \\
&= \frac{12}{3} \\
&= 4
\end{aligned}
$$

2. Consider the finite region $\mathcal{R}$ bounded by the $y = -2x$, $y = x + 6$ and the y-axis which contains the point $(-1, 3)$. Express the volume of the region $\mathcal{R}$ about the following lines using **BOTH** $\mathrm{d}x$ and $\mathrm{d}y$ integrals. You do not have to evaluate these integrals.

(a) $x = -3$, (b) $x = 1$, (c) $y = 6$, (d) $y = -1$

The two lines intersect at the point $(-2, 4)$, and the x-coordinates of the region span $[-2, 0]$, while the y-coordinates span $[0, 6]$, where $y = 0$ and $y = 6$ are where the two lines intersect the y-axis. See Figure 2 on the next page.

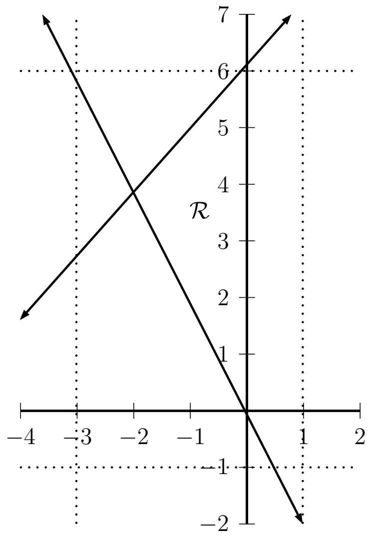

Figure 2. Graphs of the bounded region $\mathcal{R}$ and the lines of rotation.

(a) $x = -3$

First we do dx, which results in a single integral using cylindrical shells, with $r(x) = x - (-3) = x + 3$ and $h(x) = (x + 6) - (-2x) = 3x + 6$.

$$\mathcal{V} = \int_{-2}^{0} 2\pi\,(x - (-3))\,((x + 6) - (-2x))\,\mathrm{d}x = \int_{-2}^{0} 2\pi\,(x + 3)\,(3x + 6)\,\mathrm{d}x$$

To do the dy integral, we will have to break up the integral into two pieces, the first will be for $y \in [0, 4]$ and $y \in [4, 6]$ since the left function will change at $y = 4$. Also, for each point $(x, -2x)$ we have $(-y/2, y)$ and similarly for $(x, x + 6)$ we have $(y - 6, y)$. Rotating a dy slice about $x = -3$ gives a washer. For both pieces, $r_o = 3$, but for $y \in [0, 4]$, $r_i = -y/2 - (-3)$, and for $y \in [4, 6]$, $r_i = y - 6 - (-3)$. Thus,

$$\mathcal{V} = \int_{0}^{4} \pi\,\left[3^2 - (-y/2 + 3)^2\right]\,\mathrm{d}y + \int_{4}^{6} \pi\,\left[3^2 - (y - 3)^2\right]\,\mathrm{d}y$$

(b) $x = 1$

First we do dx, which results in a single integral using cylindrical shells, with $r(x) = 1 - x$ and $h(x) = (x + 6) - (-2x) = 3x + 6$.

$$\mathcal{V} = \int_{-2}^{0} 2\pi\,(1 - x)(3x + 6)\,\mathrm{d}x$$

No need to restate what must be done for the dy integral. We sill have washers, but now $r_i = 1$, and for $y \in [0, 4]$, $r_o = 1 - (-y/2) = 1 + y/2$, and for $y \in [4, 6]$, $r_o = 1 - (y - 6) = 7 - y$.

$$\mathcal{V} = \int_{0}^{4} \pi\,\left[(1 + y/2)^2 - (1)^2\right]\,\mathrm{d}y + \int_{4}^{6} \pi\,\left[(7 - y)^2 - (1)^2\right]\,\mathrm{d}y$$

(c) $y = 6$

First we do dx, which results in a single integral using the washer formula over the interval $[-2, 0]$. The outside radius will be $r_o = 6 - (-2x) = 6 + 2x$, and inside radius will be $r_i = 6 - (x + 6) = -x$.

$$\mathcal{V} = \int_{-2}^{0} \pi \left[(6+2x)^2 - (-x)^2\right] \mathrm{d}x$$

Next, we need to do the dy integral, which will once again have to be broken up. This time, we have a cylindrical shell as our object of rotation. The radius for each part will be the same, which is $r = 6 - y$. The height will be right-left, where right is always $x = 0$. For $y \in [0,4]$, left is $x = -y/2$, so $h = y/2$, and for $y \in [4,6]$, left is $x = y - 6$ giving $h = 6 - y$.

$$\mathcal{V} = \int_0^4 2\pi\,(6-y)\,(y/2)\,\mathrm{d}y + \int_4^6 2\pi\,(6-y)\,(6-y)\,\mathrm{d}y$$

(d) $y = -1$

First we do dx, which results in a single integral using the washer formula over the interval $[-2, 0]$. The outside radius will be $r_o = x + 6 - (-1) = x + 7$, and inside radius will be $r_i = -2x - (-1) = 1 - 2x$.

$$\mathcal{V} = \int_{-2}^{0} \pi \left[(x+7)^2 - (1-2x)^2\right] \mathrm{d}x$$

Next, we need to do the dy integral, which will once again have to be broken up. This time, we have a cylindrical shell as our object of rotation. The radius for each part will be the same, which is $r = y - (-1) = y + 1$. The heights do not change from part (c) − for $y \in [0,4]$, $h = y/2$, and for $y \in [4,6]$, $h = 6 - y$.

$$\mathcal{V} = \int_0^4 2\pi\,(y+1)\,(y/2)\,\mathrm{d}y + \int_4^6 2\pi\,(y+1)\,(6-y)\,\mathrm{d}y$$

3. Compute the arc length of $f(x) = \frac{2}{3}(x-1)^{3/2}$ for $x \in [1, 25]$.

First, we compute $f'(x)$:

$$f'(x) = (x-1)^{1/2} = \sqrt{x-1}.$$

Now, we plug this into the arc length formula.

$$\begin{aligned}
\mathcal{AL} &= \int_1^{25} \sqrt{1 + (f'(x))^2}\,\mathrm{d}x \\
&= \int_1^{25} \sqrt{1 + (\sqrt{x-1})^2}\,\mathrm{d}x \\
&= \int_1^{25} \sqrt{1 + x - 1}\,\mathrm{d}x \\
&= \int_1^{25} \sqrt{x}\,\mathrm{d}x \\
&= \frac{2}{3} x^{3/2} \Big|_1^{25} \\
&= \frac{2}{3}(125 - 1) \\
&= \frac{248}{3}
\end{aligned}$$