Math 2283 - Introduction to Logic
Final Exam

Assigned: 2018.04.25

Due: 2018.05.09 at 13:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire. Remember, you have two whole weeks to work on this, your masterpiece will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ___________________________________

Signature: ____

Date: ____

Definitions:

A relation R is *strongly connected* if and only if $\forall\, x, y,\ xRy \vee yRx$.

A relation S is *antisymmetric* if and only if $\forall\, x, y,\ xRy \wedge yRx \rightarrow x = y$.

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1. (15 pts) Prove that if a relation R is antisymmetric, then R' is connected.

2. (15 pts) Prove that if a relation R is reflexive, then R is not asymmetric.

3. (15 pts) Prove that if a relation R' is strongly connected, then R is asymmetric.

4. (20 pts) Let R, S, and T be arbitrary relations. Only one of the the following sentences is true. Determine, with proof, which one is true, and describe what fails in trying to prove the other sentence.

$$(R \cap S)/T \subseteq ((R/T) \cap (S/T)), \quad (R \cap S)/T \supseteq ((R/T) \cap (S/T))$$

5. (15 pts) Prove the following theorem:

$$p \rightarrow [\ (p \rightarrow q) \rightarrow q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I. $[\ p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$
Theorem II. $p \rightarrow p$

6. (20 pts) Prove the following theorem directly. (I.e. you cannot prove by the method of truth tables)

$$[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (q \vee r)$$

7. (30 pts) Define axiomatic systems (A) and (B) on a class K with relation R as follows:

Axiomatic system (A):
Axiom 1^A. The relation R is connected in the class K.
Axiom 2^A. The relation R is asymmetric in the class K.
Axiom 3^A. The relation R is transitive in the class K.

Axiomatic system (B):
Axiom 1^B. The relation R is connected in the class K.
Axiom 2^B. $(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \rightarrow \sim vRx$

Recall that Axiomatic system (A) implies that the relation R orders the class K. Prove that axiomatic systems (A) and (B) are equipollent. What does this imply about the definition of an ordering of a class K using a relation R?

8. (15 pts) Consider the following system of Axioms, where R is an arbitrary relation:

Axiom I. $\forall\, x \in \mathbf{S}\ (xRx)$
Axiom II. $\forall\, y, z \in \mathbf{S}\ (yRz \to zRy)$
Axiom III. $\forall\, x, y, z \in \mathbf{S}\ \ ((xRy \wedge yRz) \to xRz)$

Exhibit models of each of these three axioms such that:

(a) The first two sentences of the system hold, but not the last.
(b) The first and third sentence hold, but not the second.
(c) The last two sentences hold, but not the first.

Lastly, explain what this implies about the system of axioms.

9. (15 pts) Consider the relation C amongst real numbers defined as

$$xCy \overset{def}{\longleftrightarrow} x^2 + y^2 = 1$$

Determine if C is any of reflexive, irreflexive, symmetric, asymmetric, antisymmetric, transitive, intransitive, connected or strongly connected over the set of real numbers. *Remember that intransitive is defined on page 97.*

10. (20 pts) Using Axioms I – IX, Definition 1, and Theorems I – XX from pages 118 – 121, prove one of the following theorems. If you wish to prove Theorem B, then you may assume Theorem A has been proven. No extra points are given for proving both Theorems.

Theorem A: $K \cup (L \cup M) = (K \cup L) \cup M$

Theorem B: $K \cup [L \cap (M \cup N)] = [(K \cup L) \cap (K \cup M)] \cup [(K \cup L) \cap (K \cup N)]$

11. (15 pts) Construct a truth table for the sentence $[(p \vee q) \wedge (p \to r)] \to (q \vee r)$.

12. (20 pts) Consider the following definition of $\prec$:

$$x \prec y \overset{def}{\longleftrightarrow} |x| < |y|$$

Also, we will define $x \succ y \overset{def}{\longleftrightarrow} y \prec x$. The set of real numbers, together with the definitions of $\prec$ and $\succ$ do NOT constitute a model of the axiomatic system found on page 132. Determine which of the axioms hold, and which do not. Be sure to give examples where axioms do not hold. Is there a subset of the real numbers for which we do have a model for the axiomatic system?

13. (20 pts) Using the definitions of $\prec$ and $\succ$ from problem 12, determine which of the following quantified sentences are true. If a sentence is false, give a counterexample to justify your answer. You may assume that the universe of discourse is the set of real numbers.

(a) $\forall x\, \exists y\ \sim (x \prec y)$

(b) $\exists x\, \forall y\ \sim (x \prec y)$

(c) $\forall x, y\, [(x \neq y) \to ((x \succ y) \vee (x \prec y))]$

(d) $\forall x, y\, \exists z\ [(x \prec y) \to ((x \prec z) \wedge (z \prec y))]$