All About Induction

The Concept of Induction

There are three basic components to the process of induction.

(1) The base case: argue that your statement holds for the most simplest of case(s).
(2) The induction hypothesis: assume your statement holds for a specific case, usually based on a quantity n. I.e. assume $P(n)$ is true.
(3) The inductive step: show that the statement holds for the following case, using the induction hypothesis. I.e. show $P(n) \to P(n+1)$.

The Logical Framework

Induction works due to repeated applications of modus ponens. To see this, we assume that we are trying to argue that a property P holds which is based on a counting variable n, i.e. $P = P(n)$. Here n must belong to a denumerable set, and we will assume that $n = 1$ is the least element in the set for simplicity. If we know $P(1)$ is true, and we have shown that $P(n) \to P(n+1)$ is also true (for arbitrary n), then we get $P(2)$ since we have: $P(1)$ and $P(1) \to P(2)$ (setting $n = 1$ in $P(n) \to P(n+1)$) along with modus ponens yields $P(2)$. But now that we know $P(2)$ is true, letting $n = 2$ in $P(n) \to P(n+1)$ gives also $P(2) \to P(3)$, and thus by modus ponens we get $P(3)$. We can see how this process, in order, can eventually get us $P(12)$, $P(5238)$ etc...

The Cheapest of Cheap Examples

So here we go with the cheapest of cheap examples. We are going to show that the sum of the first n positive integers is $n(n+1)/2$. I.e.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

We define $P(n)$ to be as follows: 'The sum of the first n positive integers is $n(n+1)/2$'. Next, we work through the induction process, following the steps outlined above.

(1) The base case: $n = 1$ – The sum of the first (and only) positive integer 1 is 1. Letting $n = 1$ in the formula $n(n+1)/2$ yields 1 as well, thus our formula holds for the base case.

(2) The induction hypothesis: Assume that $P(n)$ is true, i.e. we assume that the sum of the first n positive integers is $n(n+1)/2$.

(3) The inductive step: We now show that $P(n+1)$ is true, using the assumption from the inductive hypothesis. To show this, we simply rewrite the sum of positive integers from 1 to $n+1$ as the sum from 1 to n plus $n+1$. I.e.

$$1 + 2 + 3 + \cdots + n + (n+1) = (1 + 2 + 3 + \cdots + n) + (n+1)$$

But we know (by assumption) that $1 + 2 + 3 + \cdots + n = n(n+1)/2$. Plugging this into the right hand side of the above equation gives

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

At this point, believe it or not, we are done! Because $P(n+1)$ is: 'The sum of the first $n+1$ positive integers is $(n+1)(n+2)/2$'. (In other words, replace n with $n+1$ everywhere in the definition of $P(n)$.) Note that we did use our inductive hypothesis to show that $P(n+1)$ was true. We can now conclude that $P(n)$ is true for all $n \geq 1$. This is induction!

A Slightly Less Cheap Example

Next we try to prove the following statement: $2^n > 2n$ for all integers $n \geq 3$. So we define $P(n)$ to be $2^n > 2n$. This is an example where the base case is not $n = 1$.

(1) The base case: Here, $n = 3$ is the base case. We simply check: $2^3 > 2 \cdot 3$. So we know $P(3)$ is true.

(2) The inductive hypothesis: We assume $P(n)$ is true, i.e. $2^n > 2n$.

(3) The inductive step: We now show that $P(n+1)$ is true, using the assumption from the inductive hypothesis. In fact, we start with the inductive hypothesis: $2^n > 2n$ and multiply both sides of the inequality by 2 (which preservers the validity of the inequality) to get

$$2 \cdot 2^n > 2 \cdot 2n$$

But $2 \cdot 2^n = 2^{n+1}$ and $2 \cdot 2n = 2(n+1)$. Plugging these into the previous inequality gives

$$2^{n+1} > 2(n+1)$$

The above inequality is exactly $P(n+1)$. So we are done!

The Concept of *Strong* Induction

The induction process outlined at the start of this document is very specific in its three steps. Induction, as defined so far, is insufficient to prove certain statemsn which rely on not just the previous case, but potentially more (or all) cases between the base case and n. The first and third components to the process of strong induction are the same as those of induction (which we now refer to as *weak* induction), so pay close attention to step (2):

(1) The base case: argue that your statement holds for the most simplest of case(s).
(2) The induction hypothesis: assume your statement holds for *all* cases starting at the base case up to the n.
(3) The inductive step: show that the statement holds for the following case, using the induction hypothesis. I.e. show $P(n) \to P(n+1)$.

The Classic Strong Induction Example

Recall that an integer k is *prime* if and only if can only be divided by itself and 1. The claim is as follows: Every positive integer greater than 1 can be factored solely into the product of prime numbers. If we tried to prove this problem via weak induction, we would not get anywhere, as there is not necessarily any relationship between the prime factors of an integer n and the next integer $n+1$ (for example, 9 and 10 share no prime factors). We will see how strong induction can be used in this situation.

Define $P(n)$ to be: 'the integer n can be expressed as the product of prime numbers'. We follow the strong induction process:

(1) The base case: $n = 2$ is the base case, and 2 is prime, thus it is already expressed as the product of prime numbers (a product of one number in this case).

(2) The induction hypothesis: we assume that $2, 3, \ldots, n$ can all be expressed as the product of prime numbers.

(3) The induction step: now we want to show that $n+1$ can be expressed as the product of prime numbers. We do not know what $n+1$ is, but we do know that either $n+1$ is prime, or it is not. So we consider both cases. If $n+1$ is prime, then it is already expressed as the product of prime numbers. If $n+1$ is not prime, then we know that there are two integers x and y such that $x \cdot y = n+1$. But $x < n+1$ and $y < n+1$, and by the inductive hypothesis, we know that x and y both can be expressed as the product of prime numbers. Thus $n+1 = x \cdot y$ is also the product of prime numbers.

How Strong Induction Works Logically

Remember that weak induction is essentially using modus ponens repeatedly after proving the base case. The idea is similar here, but we have to be careful about each step. For the base case (let's assume that it is $n = 1$ for convenience), there is no difference from that of the weak induction process. But now we assume, by the inductive hypothesis $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ and *if* we derive $P(n+1)$ from this assumption, then we have the true logical sentence

$$P(1) \wedge P(2) \wedge \cdots \wedge P(n) \to P(n+1).$$

So how does this show that $P(n)$ is true for all n greater than or equal to the base case? Well, let us look at the case $n = 2$ first. By base case, we know that $P(1)$ is true, and we also have that $P(1) \to P(2)$ by setting $n = 1$ in the above logical sentence. Thus, by modus ponens, we get $P(2)$. But now that we have $P(2)$, by the *Rule of And, Joining Together*, we also have $P(1) \wedge P(2)$. Letting $n = 3$ in the logical sentence above gives $P(1) \wedge P(2) \to P(3)$, and since we have the hypothesis of this conditional sentence, by modus ponens we get the consequent, $P(3)$. We can repeat this process as many times as desired, so $P(n)$ is true for any n greater than or equal to the base case.

A Much More Exciting Strong Induction Example

In the mathematical game Nim, there are two players and two piles of objects. At each turn, a player removes some (non-zero) number of objects from one of the piles. The player who removes the last object wins. The claim is that if the two piles contain the same number of objects at the beginning of the game, then the second player has a winning strategy which is as follows: Suppose player 1 removes n objects from one pile, then player two follows this move by removing n objects from the other pile. To prove this is a winning strategy, we will use induction.

(1) The base case: Here, $n = 1$ is our base case, which corresponds to only 1 object in each pile. If both piles contain 1 one object, the first player has only one possible move: remove the last object from one pile. The second player can then remove the last object from the other pile and thereby win.

(2) The induction hypothesis: Suppose that the second player can win any game that starts with two piles of k objects, where k is any value from 1 through n.

(3) The induction step: We need to show that the winning strategy holds for $n + 1$ objects. So, suppose that both piles contain $n + 1$ objects. A legal move by the first player involves removing j objects from one pile where $1 \leq j \leq n + 1$. The piles then contain $n + 1$ objects and $n + 1 - j$ objects. The second player can now remove j objects from the other pile. This leaves us with two piles of $n + 1 - j$ objects. If $j = n + 1$, then the second player wins as all objects have been removed from both piles. If $j < n + 1$, then we're now effectively at the start of a game with $n + 1 - j$ objects in each pile. Since $1 \leq n + 1 - j \leq n$, we know by the induction hypothesis that the second player can win the game.