Math 1513 - College Algebra

Exam #2 - 2018.10.31

Solutions

1. Below is the graph of a function $f(x)$, using the information on the graph, sketch on a separate graph the function $F(x) = -2f(x+1)+1$. Be sure to include important values on the your axes/graph, and also explain how you arrived at your graph.

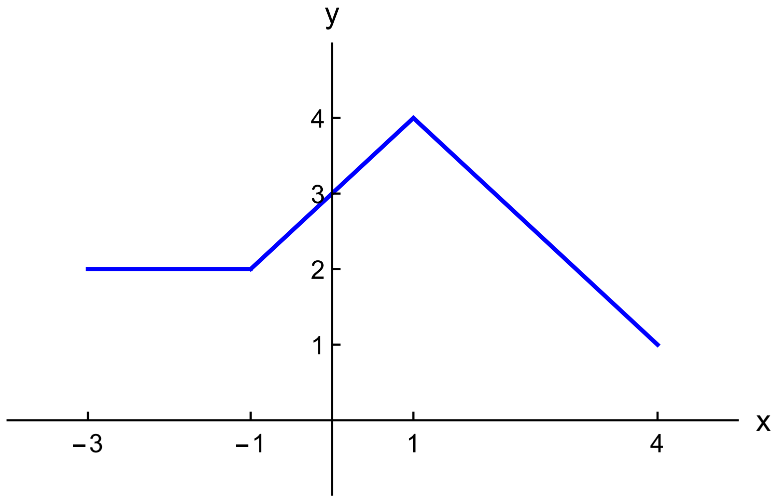

FIGURE 1. Graph of the function $f(x)$.

First, we take the blue graph and shift it to the left 1 unit, giving us the red graph. Next, we reflect about the x-axis and scale by a factor of 2, or vice versa (green). Lastly, we shift the graph up 1 unit to end up with the graph of $F(x)$ (orange).

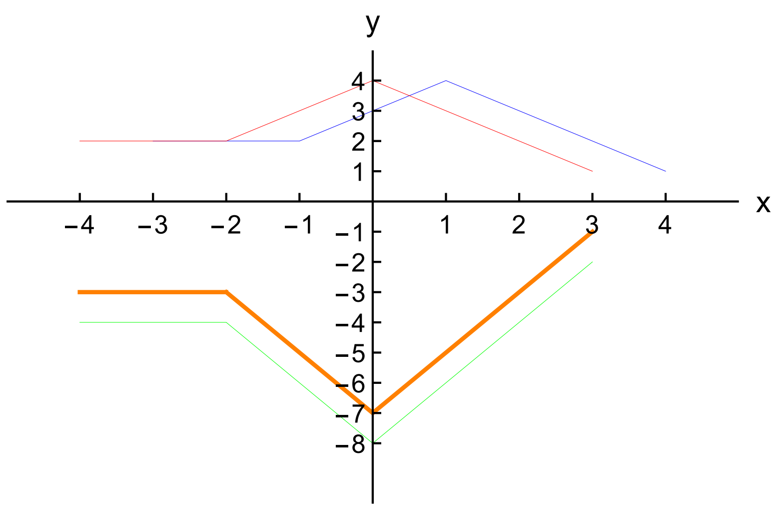

FIGURE 2. Graph of the function $F(x) = -2f(x+1)+1$.

2. Use the following two piecewise defined functions to answer each part of this problem:

$$g(x) = \begin{cases} 2, & -3 < x \leq -2 \\ 3 + x, & -1 < x < 0 \\ 2, & x = 0 \\ \left|\dfrac{x}{2} - 1\right|, & 1 \leq x \leq 4 \end{cases} \quad , \qquad h(x) = \begin{cases} x^2, & x < 0 \\ 3, & x = 0 \\ -x^2, & x > 0 \end{cases}$$

(a) Sketch the graph of $g(x)$.

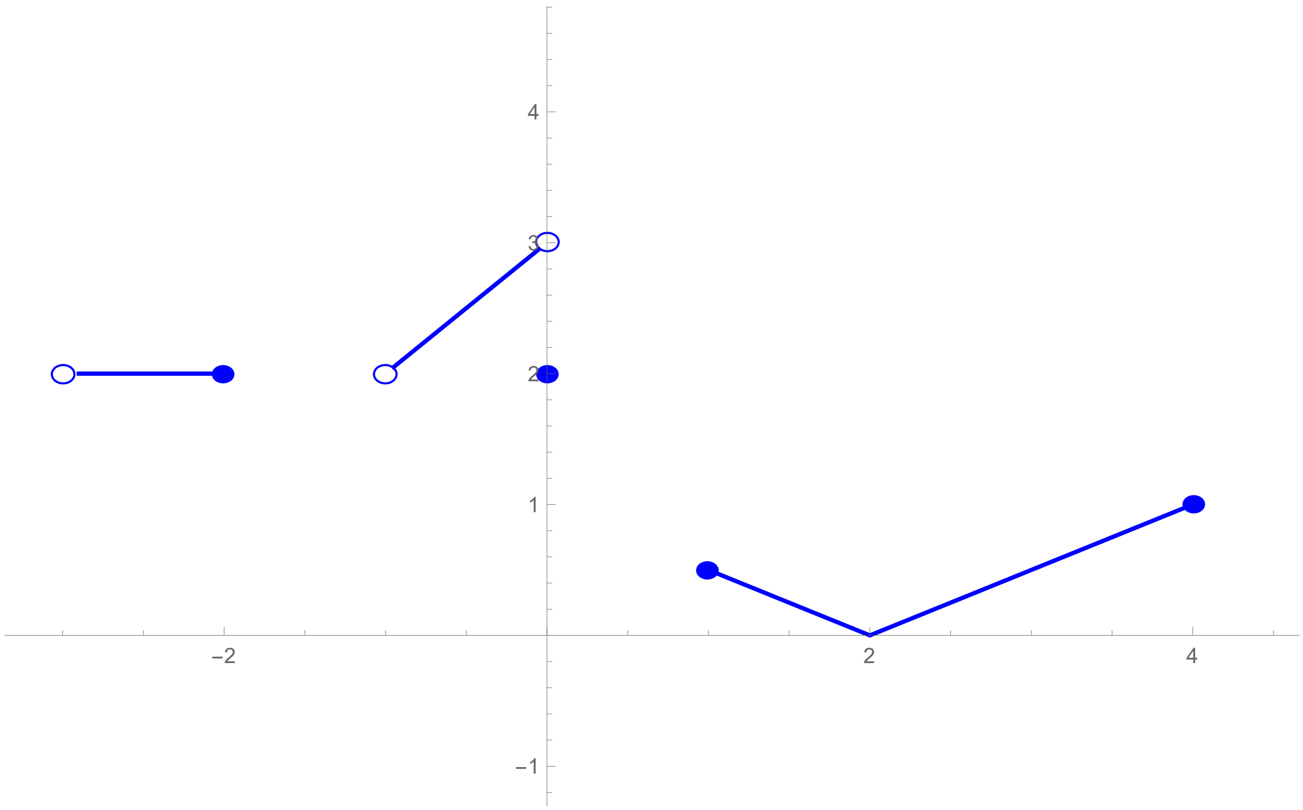

FIGURE 3. Graph of the piecewise function $g(x)$.

(b) $g(0) = 2$

(c) $g(-3)$ is undefined

(d) $h(0) = 3$

(e) $g(-2.435389) = 2$

(f) $(g \circ h)(\sqrt{2}) = g(h(\sqrt{2})) = g(-2) = 2$

(g) $(h \circ g)(-2.5) = h(g(-2.5)) = h(2) = -4$

3. For this problem, consider the two functions $k(x) = \sqrt{2x+3}$ and $m(x) = \dfrac{x+1}{x-1}$.

(a) State the domain of $k(x)$.

For $k(x)$, we require that the argument is non-negative, i.e. $2x + 3 \geq 0$ which means $x \geq -3/2$. Thus the domain is $[-3/2, \infty)$.

(b) State the domain of $m(x)$.

Since $m(x)$ is a rational function, the domain is simply everything except where the denominator is zero, which is $x = 1$. Thus the domain is $\mathbb{R} - \{1\}$.

(c) Compute the domain of $k(x) + m(x)$.

Here we take the intersection of the domains of $k(x)$ and $m(x)$. Using parts (a) and (b), we have $[-3/2, 1) \cup (1, \infty)$.

(d) Compute the domain of $k(x) \cdot m(x)$.

Nothing changes here from part (c), so the answer is exactly the same.

(e) Compute the domain of $\dfrac{m(x)}{k(x)}$.

Here we also need the added requirement that $k(x) \neq 0$, therefore, we use the answer from part (c) and remove $x = -3/2$ to get $(-3/2, 1) \cup (1, \infty)$.

4. Divide the polynomial $p(x) = x^5 + 2x^4 - 12x^3 + 15x^2 - 7x + 2$ by $q(x) = x^2 - 2x + 1$. State if the division has a remainder.

$$
\begin{array}{r}
x^3 \quad + 4x^2 - 5x + 1 \\
x^2 - 2x + 1 \,\overline{)\; x^5 + 2x^4 - 12x^3 + 15x^2 - 7x + 2} \\
\underline{-\, x^5 + 2x^4 \quad - x^3 \qquad\qquad\qquad\quad} \\
\underline{4x^4 - 13x^3 + 15x^2 \qquad\quad} \\
\underline{-\, 4x^4 \;+ 8x^3 \;\; - 4x^2 \qquad\quad} \\
-\, 5x^3 + 11x^2 - 7x \quad\; \\
\underline{5x^3 - 10x^2 + 5x \quad\;} \\
x^2 - 2x + 2 \\
\underline{-\, x^2 + 2x - 1} \\
1
\end{array}
$$

There is a remainder of 1, so $q(x)$ does not divide evenly into $p(x)$.

5. For this problem, let $p(x) = 6x^5 - 23x^4 + 3x^3 + 41x^2 - 9x - 18$.

(a) Find all the *potential* rational roots of $p(x)$ using the rational roots test.
We simply take the ratio of all factors of the constant term over the coefficient in front of the largest term. Thus, the *potential* rational roots are

$$\frac{\pm 1, 2, 3, 6, 9, 18}{\pm 1, 2, 3, 6}$$

(b) Using your answer from part (a), fully factor $p(x)$.

Note that if we plug in $x = -1$ and $x = 1$ we get roots, thus we can perform polynomial long division of $p(x)$ by $x^2 - 1$ as follows:

$$
\begin{array}{r|l}
 & 6x^3 - 23x^2 + 9x + 18 \\
\hline
x^2 - 1 & 6x^5 - 23x^4 + 3x^3 + 41x^2 - 9x - 18 \\
 & -6x^5 + 6x^3 \\
\hline
 & -23x^4 + 9x^3 + 41x^2 \\
 & 23x^4 - 23x^2 \\
\hline
 & 9x^3 + 18x^2 - 9x \\
 & -9x^3 + 9x \\
\hline
 & 18x^2 - 18 \\
 & -18x^2 + 18 \\
\hline
 & 0
\end{array}
$$

Now all we need to do is factor $q(x) = 6x^3 + 9x + 18$, whose roots must be from the list from part (a) as well. Note that $q(3) = 0$, so we can divide $q(x)$ by $x - 3$ to get

$$
\begin{array}{r|l}
 & 6x^2 - 5x - 6 \\
\hline
x - 3 & 6x^3 - 23x^2 + 9x + 18 \\
 & -6x^3 + 18x^2 \\
\hline
 & -5x^2 + 9x \\
 & 5x^2 - 15x \\
\hline
 & -6x + 18 \\
 & 6x - 18 \\
\hline
 & 0
\end{array}
$$

Now that we have a quadratic, we can apply the quadratic equation or we can test other rational roots. It is not hard to see that $6x^2 - 5x - 6 = (2x - 3)(3x + 2)$. Thus, we can fully factor $p(x)$ as $p(x) = (x-3)(x-1)(x+1)(2x-3)(3x+2)$.

6. Sketch the graph of $p(x) = -(x + 1)(x - 2)x^2(2x - 1)$.

Roots are at $x = -1$, $x = 2$, $x = 0$ and $x = 1/2$. The multiplicity of each is 1, 1, 2, and 1, respectively. The total degree is 5, which is odd, with the coefficient in from of the x^5 term being -2. Putting this all together gives

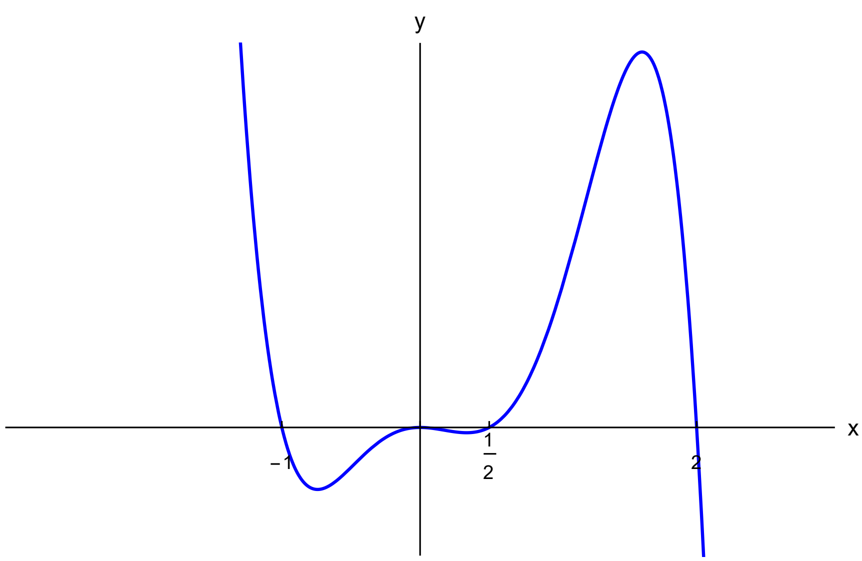

FIGURE 4. Graph of the piecewise function $p(x) = -(x + 1)(x - 2)x^2(2x - 1)$.

7. Sketch the graph of the rational function $r(x) = -\frac{1}{5}\frac{(x-2)^2(x+2)}{((x-1)(x+3))}$.

The domain for $r(x)$ is $\mathbb{R} - \{1, 3\}$, which correspond to vertical asymptotes of multiplicity 1. The roots of $r(x)$ are $x = 2, -2$, where the multiplicity of $x = 2$ is 2, and $x = -2$ is 1. To determine any horizontal/slant asymptote, we note that the degree of the denominator is 2, and the numerator is 3. Therefore we will have a slant asymptote. If we ignore that $-1/5$ out front, the numerator's terms of degree 2 or greater are: $x^3 - 2x^2$, so dividing these by x^2 gives $x - 2$. Finally, multiplying by $-1/5$ gives the slant asymptote $y = -\frac{1}{5}x + \frac{2}{5}$. The graph can now be constructed from all of this information as depicted below.

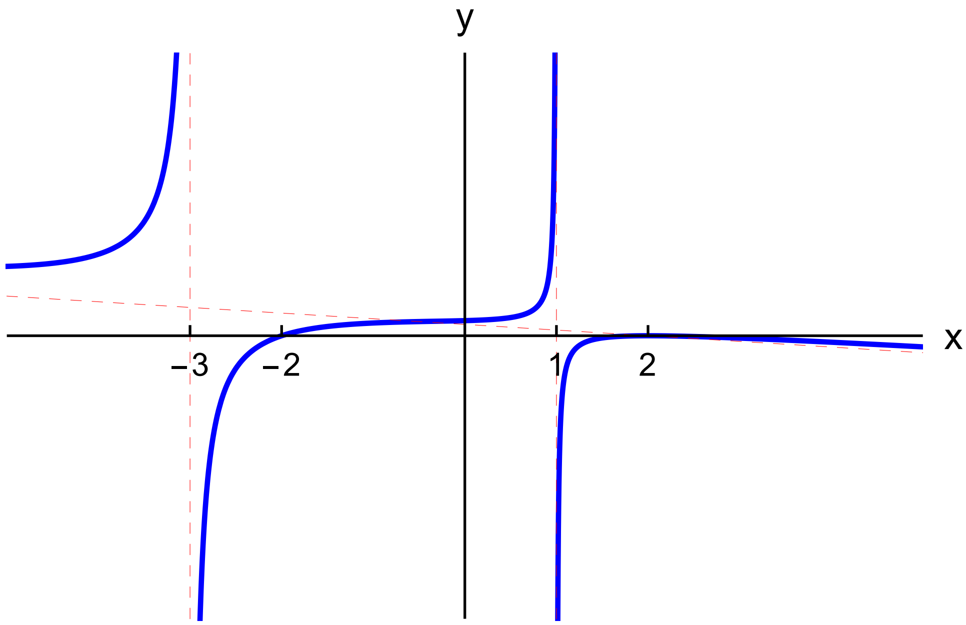

FIGURE 5. Graph of the piecewise function $r(x) = -\frac{1}{5}\frac{(x-2)^2(x+2)}{((x-1)(x+3))}$.

8. Solve the inequality $R(x) = \frac{(x+2)^2(x-5)(2x-3)}{(x+1)^2(x-1)} < 0$, express your answer in interval notation.

We consider the points where the numerator or denominator is zero. These are $x = -2$, $x = 5$, and $x = 3/2$ in the numerator, and $x = -1$ and $x = 1$ in the denominator. Ordering these, we have $-2 < -1 < 1 < 3/2 < 5$. Clearly if $x \to \infty$, $R(x) > 0$. Thus, we look at the multiplicity of the root at $x = 5$ in the numerator, which is 1. Therefore, we have that $R(x)$ becomes negative when switching from $x > 5$ to $x < 5$. Next up is $x = 3/2$, which is a root of multiplicity 1 as well, which means $R(x)$ changes sign again, to positive. So we know on the interval $(3/2, 5)$, $R(x) < 0$. Moving to $x = 1$, which is a root of multiplicity 1 in the denominator, $R(x)$ changes from positive to negative and stays that way after $x = -1$ as well since at $x = -1$, we have a root of multiplicity 2 in the denominator. Finally, at $x = -2$ we have another root of multiplicity 2, so $R(x)$ stays negative for $x < -2$. Putting all of this together, the intervals for $R(x) < 0$ are given by:

$$(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (3/2, 5)$$