Math 1613 - Trigonometry

Exam #3 - 2018.11.09

Solutions

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1. Fill out the following table completely:

$\theta°$	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
θ (rad)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$\sin(\theta)$	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
$\cos(\theta)$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{2}{4}}$	$-\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{4}{4}}$

For each of the following equations find exact solutions on the interval $[0, 2\pi)$.

2. $\csc(x) + \cot(x) = 1$

$$\begin{aligned}
\csc(x) + \cot(x) &= 1 \\
\csc(x) &= 1 - \cot(x) \\
\csc^2(x) &= 1 - 2\cot(x) + \cot^2(x) \\
1 + \cot^2(x) &= 1 - 2\cot(x) + \cot^2(x) \\
0 &= -2\cot(x)
\end{aligned}$$

We can solve the last equation above, which gives $x = \frac{\pi}{2}, \frac{3\pi}{2}$. However, $x = \frac{3\pi}{2}$ is not a solution to the original equation, so our final answer is $x = \frac{\pi}{2}$.

3. $\cos^2(\theta) = \cos(\theta) + \sin^2(\theta)$

$$\begin{aligned}
\cos^2(\theta) &= \cos(\theta) + \sin^2(\theta) \\
\cos^2(\theta) &= \cos(\theta) + 1 - \cos^2(\theta) \\
2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \\
(\cos(\theta) - 1)(2\cos(\theta) + 1) &= 0
\end{aligned}$$

By the product of zeros rule, we have $\cos(\theta) - 1 = 0$ or $2\cos(\theta) + 1 = 0$. Solving in each gives $\cos(\theta) = 1$ and $\cos(\theta) = -\frac{1}{2}$, which gives $\theta = 0$ for the first equation, and $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

For each of the following equations find exact solutions on the interval $[0, 2\pi)$.

4. $\dfrac{\sin(2t) + \sin(4t)}{\cos(2t) - \cos(4t)} = \cot(t)$

First, we use the fact that $\sin(2t) + \sin(4t) = 2\sin(3t)\cos(t)$ and $\cos(2t) - \cos(4t) = -2\sin(3t)\sin(-t)$ by the sum-to-product identities for $\sin(A) + \sin(B)$ and $\cos(A) - \cos(B)$. Thus

$$\begin{aligned}\frac{\sin(2t)+\sin(4t)}{\cos(2t)-\cos(4t)} &= \frac{2\sin(3t)\cos(t)}{-2\sin(3t)\sin(-t)} \\ &= \frac{2\sin(3t)\cos(t)}{2\sin(3t)\sin(t)} \\ &= \frac{\cos(t)}{\sin(t)} \\ &= \cot(t)\end{aligned}$$

5. $\sin(x)\sin(y)\sin(z) = \frac{1}{4}\left[\sin(x+y-z) + \sin(y+z-x) + \sin(z+x-y) - \sin(x+y+z)\right]$

We start with the left side and use the $\sin(A)\sin(B)$ identity on $\sin(x)\sin(y)$ on the left side of the above equation.

$$\begin{aligned}\sin(x)\sin(y)\sin(z) &= [\sin(x)\sin(y)]\sin(z) \\ &= \frac{1}{2}[\cos(x-y) - \cos(x+y)]\sin(z) \\ &= \frac{1}{2}[\cos(x-y)\sin(z) - \cos(x+y)\sin(z)] \\ &= \frac{1}{2}\left[\frac{1}{2}[\sin(x-y+z) - \sin(x-y-z)] - \frac{1}{2}[\sin(x+y+z) - \sin(x+y-z)]\right] \\ &= \frac{1}{4}[\sin(x-y+z) - \sin(x-y-z) - \sin(x+y+z) + \sin(x+y-z)] \\ &= \frac{1}{4}[\sin(x-y+z) - \sin(x-y-z) - \sin(x+y+z) + \sin(x+y-z)] \\ &= \frac{1}{4}[\sin(x+y-z) - \sin(x-y-z) + \sin(x-y+z) - \sin(x+y+z)] \\ &= \frac{1}{4}[\sin(x+y-z) - \sin(-((y+z)-x)) + \sin(z+x-y) - \sin(x+y+z)] \\ &= \frac{1}{4}[\sin(x+y-z) + \sin(y+z-x) + \sin(z+x-y) - \sin(x+y+z)]\end{aligned}$$

6. Write $\sin(2\cot^{-1}(x))$ as an algebraic expression only, free of trigonometric or inverse trigonometric functions.

We first use the double angle identity $\sin(2A) = 2\sin(A)\cos(A)$ to get

$$\sin(2\cot^{-1}(x)) = 2\sin(\cot^{-1}(x))\cos(\cot^{-1}(x))$$

Setting $\theta = \cot^{-1}(x)$ gives $\cot(\theta) = x$, which gives a triangle with adjacent side of length x, opposite side length 1, and hypotenuse length $\sqrt{1+x^2}$. With this triangle and given angle θ gives $\sin(\theta) = 1/\sqrt{1+x^2}$ and $\cos(\theta) = x/\sqrt{1+x^2}$. Thus

$$\begin{aligned}\sin(2\cot^{-1}(x)) &= 2\sin(\cot^{-1}(x))\cos(\cot^{-1}(x)) \\ &= 2\sin(\theta)\cos(\theta) \\ &= 2\frac{1}{\sqrt{1+x^2}}\frac{x}{\sqrt{1+x^2}} \\ &= 2\frac{x}{1+x^2}\end{aligned}$$

7. Solve the following equation over the interval $[0, 2\pi)$: $\sin\left(\frac{5}{2}x\right) = -\frac{\sqrt{3}}{2}$.

First note that $\sin(\theta) = -\frac{\sqrt{3}}{2}$ at $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$ on the interval $[0, 2\pi)$. Thus, we solve two equations:

$$\frac{5}{2}x = \frac{4\pi}{3} + 2\pi k, \text{ or } \frac{5}{2}x = \frac{5\pi}{3} + 2\pi k,$$

for k any integer. Solving for x in each gives

$$x = \frac{8\pi}{15} + \frac{4\pi}{5}k = \frac{\pi(8 + 12k)}{15}, \text{ or } x = \frac{2\pi}{3} + \frac{4\pi}{5}k = \frac{\pi(10 + 12k)}{15}$$

In each of these, we can let $k = 0$ or $k = 1$. So our final solution set is

$$x = \frac{8\pi}{15}, \frac{20\pi}{15}, \frac{10\pi}{15}, \frac{22\pi}{15}.$$