

Math 2283 - Introduction to Logic Final Exam

Assigned: 2018.11.26

Due: 2018.12.10 at 08:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire. Remember, you have two whole weeks to work on this, your masterpiece will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

“I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.”

Printed Name: _____

Signature: _____

Date: _____

Part I

Definitions:

A relation R is *strongly connected* if and only if $\forall x, y, xRy \vee yRx$.

A relation S is *antisymmetric* if and only if $\forall x, y, xRy \wedge yRx \rightarrow x = y$.

1. (15 pts) Prove that if a relation R is antisymmetric, then R' is connected.
2. (15 pts) Prove that if a relation R is reflexive, then R is not asymmetric.
3. (15 pts) Prove that if a relation R' is strongly connected, then R is asymmetric.
4. (30 pts) Define axiomatic systems (A) and (B) on a class K with relation R as follows:

Axiomatic system (A) :

Axiom 1^A . The relation R is connected in the class K .

Axiom 2^A . The relation R is asymmetric in the class K .

Axiom 3^A . The relation R is transitive in the class K .

Axiomatic system (B) :

Axiom 1^B . The relation R is connected in the class K .

Axiom 2^B . $(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \rightarrow \sim vRx$

Recall that Axiomatic system (A) implies that the relation R orders the class K . Prove that axiomatic systems (A) and (B) are equipollent. What does this imply about the definition of an ordering of a class K using a relation R ?

5. (15 pts) From the midterm, remember that we defined the connective \downarrow by the following fundamental truth table:

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Furthermore, we had $\sim p \longleftrightarrow (p \downarrow p)$ and $(p \wedge q) \longleftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$. Express $p \vee q$, $p \rightarrow q$, and $p \leftrightarrow q$ in terms of only the \downarrow connective.

6. (15 pts) Prove for any three arbitrary relations R , S , and T , that $(R \cup S)/T = (R/T) \cup (S/T)$.
7. (15 pts) Given the set $\mathbb{K} = \{0, 1, 2, 3, 4\}$, define two distinct operations, \oplus and \otimes , on the set \mathbb{K} such that \mathbb{K} is an Abelian group with respect to \oplus , and also with respect to \otimes . Your definitions of the operations \oplus and \otimes must be distinct when applied to \mathbb{K} .

Part II

Propositional System Ξ :

Symbols

The system uses lower case letters such as p , q , and r , to represent propositional symbols.

The symbol \vee is the disjunctive connective: $p \vee q$ means 'p or q'.

$\bar{\quad}$ is the symbol for negation: \bar{p} is the negation of sentence p , while $\overline{p \vee q}$ is the negation of the sentence $p \vee q$.

Well Formed Formulas (WFFs)

- (1) Any propositional symbol is a wff.
- (2) If A is a wff, then \bar{A} is a wff.
- (3) If A and B are wffs, then $A \vee B$ is a wff.
- (4) Nothing else is a wff.

Examples:

- (1) p is a wff.
- (2) $\bar{p} \vee q$ is a wff.
- (3) $\overline{p \vee q \vee r}$ is a wff.
- (4) $\overline{p \vee q} \vee r$ is not a wff.

Definition: A wff A is a *primitive disjunction* iff A is of the form $B_1 \vee B_2 \vee \dots \vee B_n$ where each B_i is either a propositional symbol or a propositional symbol with a bar over it (i.e. each B_i is of the form p or \bar{p}).

Examples:

- (1) $p \vee \bar{q} \vee r$ is a primitive disjunction.
- (2) $\bar{p} \vee \bar{q}$ is not a primitive disjunction.
- (3) $\overline{\bar{p} \vee q}$ is not a primitive disjunction.
- (4) $\bar{p} \vee \bar{q} \vee q \vee p$ is a primitive disjunction.

Definition: Every wff A is a *disjunctive part* of itself, and if A is a *disjunctive part* of $B \vee C$, then B is a *disjunctive part* of A and so is C .

Examples:

- (1) p is a disjunctive part of p .
- (2) \bar{p} is a disjunctive part of $q \vee \bar{p} \vee p$.
- (3) $\bar{p} \vee p$ is a disjunctive part of $q \vee \bar{p} \vee p$.
- (4) $q \vee p$ is a not disjunctive part of $q \vee \bar{p} \vee p$.

Notation: $D(A)$ is a wff of which A is a disjunctive part, and $D(B)$ is the result of replacing *one occurrence* of the disjunctive part A in $D(A)$ by B .

Examples:

- (1) If $D(p)$ is $p \vee \bar{p} \vee q$, then $D(q \vee \bar{r})$ is $q \vee \bar{r} \vee \bar{p} \vee q$.
- (2) If $D(q)$ is $p \vee \bar{p} \vee q$, then $D(q \vee \bar{r})$ is $p \vee \bar{p} \vee q \vee \bar{r}$.
- (3) If $D(\bar{p})$ is $p \vee \bar{p} \vee q$, then $D(q \vee \bar{r})$ is $p \vee q \vee \bar{r} \vee q$.
- (4) If $D(p)$ is $p \vee \bar{p} \vee p \vee r$, then $D(\bar{r})$ is not $\bar{r} \vee \bar{p} \vee \bar{r} \vee r$.
- (5) If $D(p)$ is $p \vee \bar{p} \vee p \vee r$, then $D(\bar{r})$ is $p \vee \bar{p} \vee \bar{r} \vee r$.
- (6) If $D(p)$ is $p \vee \bar{p} \vee p \vee r$, then $D(\bar{r})$ is $\bar{r} \vee \bar{p} \vee p \vee r$.

Axioms of System Ξ :

There are an infinite number of axioms for System Ξ . A wff A is an axiom iff A is a primitive disjunction and for some propositional symbol B , both B and \overline{B} are disjunctive parts of A .

Examples:

- (1) $q \vee \overline{r} \vee \overline{q} \vee p$ is an axiom.
- (2) $q \vee \overline{r} \vee q \vee \overline{r}$ is not an axiom.
- (3) $\overline{q} \vee \overline{q}$ is not an axiom.
- (4) $q \vee \overline{r} \vee \overline{q} \vee q \vee \overline{r}$ is not an axiom.
- (5) $\overline{q} \vee r \vee p \vee \overline{q} \vee \overline{r}$ is an axiom.

Rules of Inference for System Ξ

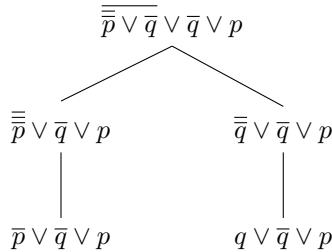
Rule I: $D(\overline{\overline{A}})$ is an immediate consequence of $D(A)$.

Rule II: $D(A \vee B)$ is an immediate consequence of $D(\overline{A})$ and $D(\overline{B})$

Examples:

- (1) $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
- (2) $p \vee \overline{\overline{r}} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
- (3) $\overline{\overline{p}} \vee \overline{q} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ and $\overline{q} \vee \overline{r} \vee \overline{p} \vee q$ by Rule II.
- (4) $q \vee \overline{\overline{p}} \vee \overline{q} \vee \overline{p} \vee r$ is an immediate consequence of $q \vee \overline{p} \vee \overline{p} \vee r$ and $q \vee \overline{q} \vee \overline{p} \vee r$ by Rule II.

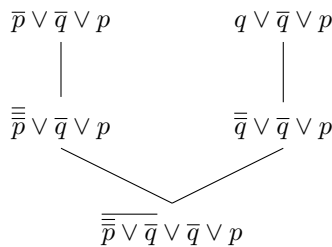
There exists a method of proof in System Ξ to derive any tautological sentence which can be aided by the construction of a tree. An example of this method is given in the two trees below for the sentence $\overline{\overline{p}} \vee \overline{q} \vee \overline{q} \vee p$. Note that the trees are simply reflections of each other, one growing downwards, the second growing upwards.



The first tree (which grows downwards) is used to dissect the original sentence using Rules I and II and the axioms given. Each branch of the tree terminates at an axiom. Notice, in the lowest branch of the above tree, the two sentences $\overline{p} \vee \overline{q} \vee p$ and $q \vee \overline{q} \vee p$ are axioms, as the first contains both p and \overline{p} , while the second contains both q and \overline{q} .

To build the tree, work from left to right and pick the leftmost portion of the wff which is not in primitive disjunctive form. For instance, $\overline{\overline{p}} \vee \overline{q}$ is the only portion of the original sentence not in primitive disjunctive form. Only an application of Rule II would yield $\overline{\overline{p}} \vee \overline{q}$ as part of a wff, so we simply determine the two wffs so that when Rule II is applied to them, we get $\overline{\overline{p}} \vee \overline{q}$. The two wffs required are $\overline{\overline{p}} \vee \overline{q} \vee p$ and $\overline{q} \vee \overline{q} \vee p$.

At each stage, we check to see if the wffs used are primitive disjunctions. In this example, they are not, but it can be easily seen that Rule I can be used on $\overline{\overline{p}} \vee \overline{q} \vee p$ to arrive at $\overline{p} \vee \overline{q} \vee p$, and can also be used on $q \vee \overline{q} \vee p$ to arrive at $\overline{q} \vee \overline{q} \vee p$. Now every branch in the tree above ends in a primitive disjunction. Furthermore, they are axioms and thus we can build a proof to generate the sentence in question from axioms using only Rules I and II. To make it easier to see this, we invert the tree as below and build the proof.



The following is the proof generated by the tree given above. Note that only axioms, and Rules I and II are used. This is the approach to be taken for any wff in this system which you believe is a tautology.

- | | | |
|-----|--|--------------------------------|
| (1) | $q \vee \bar{q} \vee p$ | Axiom |
| (2) | $\bar{p} \vee \bar{q} \vee p$ | Axiom |
| (3) | $\bar{\bar{q}} \vee \bar{q} \vee p$ | Rule I applied to (1) |
| (4) | $\bar{\bar{\bar{p}}} \vee \bar{q} \vee p$ | Rule I applied to (2) |
| (5) | $\bar{\bar{\bar{\bar{p}}}} \vee \bar{q} \vee \bar{q} \vee p$ | Rule II applied to (4) and (3) |
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1. (15 pts) Explain why the conditions stated to be an axiom of System Ξ imply that every axiom is a tautology.

2. (30 pts) Construct arguments that Rules I and II of System Ξ indeed preserve the notion of a tautology. I.e. if Rule I is applied to a tautological sentence A , then the result is also a tautological sentence. Similarly, if Rule II is applied to two tautological sentences A and B , the result is also a tautology. You do not need a rigorous proof, but your argument should be convincing.

3. (15 pts) Is System Ξ consistent? Explain your answer thoroughly.

4. (15 pts) Is System Ξ complete? Explain your answer thoroughly.

5. (10 pts) Express the logical sentence $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$ as a valid sentence of System Ξ . I.e. write it in terms of negations and disjunctions in such a way that it is a wff of System Ξ .

6. (10 pts) Create a tree for the wff you constructed in the previous problem.

7. (10 pts) Using the tree from the previous problem, construct a proof using only axioms, and Rules I and II, that shows your wff is a tautology.

8. (30 pts) Starting with the sentence $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$, repeat the work done in problems 5, 6, and 7. I.e. convert the Law of Hypothetical Syllogism to a wff of System Ξ and prove it is a tautology in the same manner as you did for $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$.