Metalogic Cheat Sheet

The formal language *P*:

Six symbols: p, $'$, $\sim$, $\supset$, (,)
Propositional symbols of P: p followed by one or more $'$, e.g. p', p'', p'''
Formulas (wffs) of P: valid sentences in the formal language P satisfy the following criteria:
(1) A single propositional symbol of P.
(2) If A is a formula, the $\sim A$ is a formula.
(3) If A and B are formulas, then $(A \supset B)$ is a formula (parentheses are sometimes dropped if $A \supset B$ is alone).

Definition: An **interpretation** of P is an assignment to each propositional symbol of P a T or F, but not both.

Definition: I is a **true interpretation** of P if for any formulas A and B
(1) If A is a propositional formula, the A is true for I iff I assigns a value of T to A
(2) $\sim A$ is true for I iff A is not true for I
(3) $A \supset B$ is true for I iff either A is not true or B is true for I

Definition: A **false interpretation** of P: A is false for I iff A is not true for I

Definition: An interpretation I is a **model** of a set of formulas of P iff every formula in the set is true for I.

Definition: ($\models_\mathbf{P}$ **A**) A is a **logically valid formula** of P iff A is true for every interpretation I of P.

Definition: A **model-theoretically consistent** set of formulas of P is one that has a model.

Definition: A **model-theoretically inconsistent** set of formulas of P is one that has no model.

Definition: (**A** $\models_\mathbf{P}$ **B**) A formula B of P is a **semantic consequence of a formula** A **of *P*** iff there is no interpretation of P for which A is true and B is false.

Definition: ($\mathbf{\Gamma} \models_\mathbf{P}$ **B**) A formula B of P is a **semantic consequence of a set of formulas** Γ **of *P*** iff there is no interpretation of P for which every formula in Γ is true and B is false.

19.6 If $\models_P A$ and $\models_P A \supset B$, then $\models_P B$
19.7 $A \models_P B$ iff $\models_P A \supset B$

Definition: A is a **tautology** of P iff A is true for every interpretation of P iff A is a logically valid formula of P.

For Statements 20.1-20.5, A and B are arbitrary formulas of P while Γ and Δ are arbitrary sets of formulas of P.
20.1 $A \models_P A$
20.2 If $\Gamma \models_P A$, then $\Gamma \cup \Delta \models_P A$
20.3 If $\Gamma \models_P A$ and $A \models_P B$, then $\Gamma \models_P B$
20.4 If $\Gamma \models_P A$ and $\Gamma \models_P A \supset B$, then $\Gamma \models_P B$
20.5 If $\models_P A$ then $\Gamma \models_P A$

20.6 **The Interpolation Theorem for *P***: If $\models_P A \supset B$ and A and B have at least one propositional symbol in common, then there is a formula C of P all of whose propositional symbols occur in both A and B such that $\models_P A \supset C$ and $\models_P C \supset B$.

The Deductive Apparatus for $\boldsymbol{P}$ – The Propositional System:

Definition: Formal language and a deductive apparatus is known as a **Formal System**

Axiom Schemas of the Propositional System (PS)
[**PS 1**] $A \supset (B \supset A)$
[**PS 2**] $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
[**PS 3**] $(\sim A \supset \sim B) \supset (B \supset A)$

Rule of Inference of (PS)
If A and B are any formulas of P, then B is an **immediate consequence** in PS of the pair of formulas A and $A \supset B$.

Definition: A **proof in $\boldsymbol{PS}$** is a finite string of formulas of P each one of which is an axiom of PS or an immediate consequence of two formulas preceding it in the string.

Definition: ($\vdash_{\mathbf{PS}} \mathbf{A}$) A formula A is a theorem of PS iff there is some proof in PS whose last formula is A.

Definition: A string of formulas is a **derivation** in PS of a wff A from a set Γ of wffs of P iff
(1) it is a finite string of formulas of P
(2) the last formula in the string is A
(3) each formula of the string is one of the following three
(i) an axiom of PS
(ii) an immediate consequence of two formulas preceding it in the string
(iii) an element of the set Γ

Remark: Every proof is a derivation, but not all derivations are proofs.

Definition: ($\boldsymbol{\Gamma} \vdash_{\mathbf{PS}} \mathbf{A}$) A formula A is a **syntactic consequence** in PS of a set Γ of formulas of P iff there is a derivation in PS of A from the set Γ.

Definition: A set Γ if formulas of P is a **proof-theoretically consistent (p-consistent)** set of PS iff for no formula A of P that we have both $\vdash_{\mathbf{PS}} \mathbf{A}$ and $\vdash_{\mathbf{PS}} \sim \mathbf{A}$.

Definition: A set Γ if formulas of P is a **proof-theoretically inconsistent (p-inconsistent)** set of PS iff there exists a formula A of P such that $\vdash_{\mathbf{PS}} \mathbf{A}$ and $\vdash_{\mathbf{PS}} \sim \mathbf{A}$.

Remark: Model-theoretically consistency is bases solely on form, whereas proof-theoretically consistency relies on the specific rules of inference (such as the rule of detachment).

For Statements 23.1-23.5, A and B are arbitrary formulas of P while Γ and Δ are arbitrary sets of formulas of P.
23.1 $A \vdash_{PS} A$
23.2 If $\Gamma \vdash_{PS} A$, then $\Gamma \cup \Delta \vdash_{PS} A$
23.3 If $\Gamma \vdash_{PS} A$ and $A \vdash_{PS} B$, then $\Gamma \vdash_{PS} B$
23.4 If $\Gamma \vdash_{PS} A$ and $\Gamma \vdash_{PS} A \supset B$, then $\Gamma \vdash_{PS} B$
23.5 If $\vdash_{PS} A$ then $\Gamma \vdash_{PS} A$
23.6 $\vdash_{PS} A$ iff $\{\ \} \vdash_{PS} A$
23.7 $\Gamma \vdash_{PS} A$ iff there is a finite subset Δ of Γ such that $\Delta \vdash_{PS} A$.

Definition: A system S is **simply consistent** iff for no formula A of S are both A and the negation of A theorems of S.

Definition: A system S is **absolutely consistent** iff at least one formula of S is not a theorem of S.

26.1 **The Deductive Theorem for PS**: If $\Gamma, A \vdash_{PS} B$, then $\Gamma \vdash_{PS} A \supset B$

28.1 Every Axiom of *PS* is logically valid.
28.2 The Rule of Inference preserves logical validity.
28.3 Every theorem of *PS* is logically valid, i.e. if $\vdash_{PS} A$ then $\models_P A$
28.4 Syntactic consequence in *PS* implies semantic consequence in *P*, i.e. if $\Gamma \vdash_{PS} A$ then $\Gamma \models_P A$
28.5 Modus Ponens for $\supset$ preserves truth-for-an interpretation *I*.

Definition: A formal system *S* with language *L* is **complete with respect to the class of all truth-functional tautologies** iff (1) *L* is adequate for the expression of any truth function and (2) every tautology of *L* is a theorem of *S*.

Definition: *PS* is **complete with respect to the class of all truth-functional tautologies** iff (1) *P* is adequate for the expression of any truth function and (2) every tautology of *P* is a theorem of *PS*.

Definition: *PS* is **semantically complete** iff every logically valid formula of *P* is a theorem of *PS*, i.e. iff if $\models_P A$ then $\vdash_{PS} A$.

Definition: *PS* is **syntactically complete** iff no unprovable schema can be added to it as an axiom-schema without inconsistency.

31.15 **The Semantic Completeness Theorem for *PS***: Every logically valid formula of *P* is a theorem of *PS*. I.e. $\models_P A$ then $\vdash_{PS} A$.

32.6 If a set of formulas of *P* has a model, then it is a p-consistent set of *PS*.
32.7 $\Gamma \cup \{\sim A\}$ is a p-inconsistent set of *PS* iff $\vdash_{PS} A$.
32.8 $\Gamma \cup \{A\}$ is a p-inconsistent set of *PS* iff $\vdash_{PS} \sim A$.

Definition: Γ is a **maximally p-consistent set of *PS*** iff Γ is a p-consistent set of *PS* and if *A* is an arbitrary formula of *P*, then either *A* is a member of Γ of $\Gamma, A \vdash_{PS} B$ and $\Gamma, A \vdash_{PS} \sim B$ for some formula *B* of *P*.

32.9 For any maximally p-consistent set Γ of *PS* and any formula *A* of *P*, exactly one of *A* and $\sim A$ is in Γ.
32.10 For any maximal p-consistent set Γ of *PS* and any formula *A* of *P*, if $\Gamma \vdash_{PS} A$ then *A* is a member of Γ.
32.11 **Enumeration Theorem for *P***: The formulas of *P* are effectively enumerable.
32.12 **Lindenbaum's Lemma for *PS***: A p-consistent set of *PS* is a subset of some maximal p-consistent set of *PS*.
32.13 Every p-consistent set of *PS* has a model.
32.14 **The Strong Completeness Theorem for *PS***: If $\Gamma \models_P A$ then $\Gamma \vdash_{PS} A$.
32.15 **The Semantic Completeness Theorem for *PS***: If $\models_P A$ then $\vdash_{PS} A$.
32.16 A set of formulas of *P* is a p-consistent set of *PS* iff it has a model.
32.17 $\models_P A$ iff $\vdash_{PS} A$.
32.18 **The Finiteness Theorem for *P***: $\Gamma \models_P A$ iff there is a finite subset Δ of Γ such that $\Delta \models_P A$.
32.19 If every finite subset of a set Γ of formulas of *P* is a p-consistent set of *PS*, then Γ is a p-consistent set of *PS*.
32.20 **The Compactness Theorem for *P***: If every finite subset of a set Γ of formulas of *P* has a model, then Γ has a model.
32.21 **The Interpolation Theorem for *PS***: If $\vdash_{PS} A \supset B$ and *A* and *B* have at least one propositional symbol in common, then there is a formula *C* of *P* all of whose propositional symbols occur in both *A* and *B* such that $\vdash_{PS} A \supset C$ and $\vdash_{PS} C \supset B$.

Definition: A formal system S is **complete** iff for each formula A (of the language system) either A or $\sim A$ is a theorem of S.
Definition: PS is **syntactically complete** (in one sense) iff no unprovable schema can be added to it as an axiom-schema without inconsistency.

33.1 PS is syntactically complete.

33.2 If $\sim A$ is any formula of P that is not a theorem of PS, then A can consistently added to PS as an axiom.

Definition: A system S is **decidable** iff there is an effective method for telling, for each formula of S, whether or not it is a theorem of S.

34.1 PS is decidable.

Definition: A formula A is **decidable in a system *S*** iff either A or $\sim A$ is a theorem of S.
Definition: A system S has an **effective proof procedure** iff, given an arbitrary theorem T of S, there is an effective method for constructing a proof in S of T.

36.1–3 The three axioms of PS are mutually independent.

The formal quantified language Q (and Q^+):

Twelve symbols: p, $'$, x, a, f, F, *, $\sim$, $\supset$, $\bigwedge$, (,)
Proposition symbols: $p', p'', \ldots$
Individual variables (variables for short): $x', x'', \ldots$
Individual constants (constants for short): $a', a'', \ldots$
Function symbols: $f^{*\prime}, f^{*\prime\prime}, \ldots, f^{**\prime}, f^{**\prime\prime} \ldots$ A function symbol with n-asterisks is an **n-place function symbol**.
Predicate symbols: $F^{*\prime}, F^{*\prime\prime}, \ldots, F^{**\prime}, F^{**\prime\prime} \ldots$ A predicate symbol with n-asterisks is an **n-place predicate symbol**. (predicate symbol is something which has a truth value).
Connectives: $\sim$, $\supset$
Universal Quantifier: $\bigwedge$
Brackets: (,)
Definition: An expression is a **term** iff it is an individual constant, an individual variable, or an n-place function followed by n terms.
Definition: A term is **closed** iff no variable occurs in it.
Formulas (wffs) of Q (or Q^+): valid sentences must satisfy the following criteria:

(1) Any propositional symbol is a wff and an atomic wff.

(2) If F is an n-place predicate symbol and $t_1, \ldots, t_n$ are terms (not necessarily distinct), then $Ft_1, \ldots, t_n$ is a wff and an atomic wff.

(3) If A is a wff and v is an individual variable, then $\bigwedge vA$ is a wff.

(4) If A is a wff, the $\sim A$ is a wff.

(5) If A and B are wffs, then $(A \supset B)$ is a wff.

Definition: In $\bigwedge vA$ if A is a wff then A is the **scope** of the quantifier $\bigwedge$.
Definition: An occurrence of a variable v is **bound** in a wff iff either it immediately follows a quantifier in the wff, or it is within the scope of a quantifier that has v as its variable.
Definition: An occurrence of a variable v is **free** iff it is not bound.
Definition: A wff A is **atomic** iff A is a wff without quantifiers.
Definition: **At/v** is the wff obtained from A by substituting t for all free occurrences of v in A.
Definition: t **is free for** v **in** A if

(1) if t is a variable, then t occurs free in At/v,

(2) if t is a term in which any variables occur, then wherever t is substituted for free occurrences of v in A all occurrences of variables in t remain free.

Definition: If t is a **closed term**, then t is free for v in any formula A.
Definition: A **closed** wff is a wff in which there are no free occurrences of any variable.
Definition: A wff is **open** iff it is not closed.

Closures:

(1) If A is a wff in which the variables $v_1, \ldots, v_n$ have free occurrences, then A preceded by $\bigwedge v_1 \cdots \bigwedge v_n$ is a closure of A.

(2) If A is a sentence (closed wff), then A is the closure of A and so is $\bigwedge vA$ where v is any variable.

(3) Any closure of a closure of A is a closure of A.

Definition: An **arbitrary closure** of A is denoted A^c.
Definition: The **existential(particular) quantifier** $\bigvee$ is defined as $\bigvee vA \longleftrightarrow \sim \bigwedge v \sim A$.

An *interpretation* of Q (or Q^+):

An *interpretation* of Q (or Q^+) consists in the specification of some non-empty set (*domain*) and the following assignments:
(1) Each propositional symbol gets a T or F.
(2) Each individual constant gets a fixed value from the domain D.
(3) Each function symbol is explicitly defined and variables are evaluated from values of the domain D.
(4) Each predicate symbol is explicitly defined (as a relation or property among objects in the domain D).
(5) Connectives are interpreted as defined by their fundamental truth values.
(6) Quantifiers refer explicitly to constants from the domain D.

Definition: A **sequence of n terms** s is ordered list (sequence) of objects, usually from the domain D.

Semantics for Q (or Q^+):

The satisfaction of a formula A by a sequence s (for a given interpretation I):
Here I is an interpretation, D is the domain, and s is an arbitrary sequence whose terms are members of D.
(For (1)–(4), we do not actually have to look at s, since there are no variables, in order to determine the truth value.)

(1) If A is a propositional truth symbol, then s satisfies A iff I assigns the truth value of T to A.
(2) If A is of the $\sim B$, then s satisfies A iff s does not satisfy B.
(3) If A is of the form $B \supset C$, then s satisfies A iff either s does not satisfy B or s does satisfy C.
(4) If A is a closed atomic wff without function symbols and not a propositional symbol, then it is of the form $Fc_1 \ldots c_n$, where F is an n-place predicate symbol and $c_1, \ldots, c_n$ are individual constants. If $c_1, \ldots, c_n$ are assigned values $d_1, \ldots, d_n$ from D, then s satisfies A iff the ordered n-tuple $\langle d_1, \ldots, d_n \rangle$ is a member of the set of ordered n-tuples assigned by I to the predicate symbol F.
(5) If A is an atomic wff of the form $Fv_1, \ldots, v_n$, F an n-place predicate symbol, $v_1, \ldots, v_n$ variables. The ith variable gets assigned the ith entry in the sequence s, thus $v_1, \ldots, v_n$ are assigned $\langle d_1, \ldots, d_n \rangle$ from s. The sequence s satisfies A iff $\langle d_1, \ldots, d_n \rangle$ is a member of the set of ordered n-tuples assigned by I to the predicate symbol F.
(6) If A is an atomic wff of the form $Ft_1, \ldots, t_n$, F an n-place predicate symbol, $t_1, \ldots, t_n$ terms without function symbols. Here t_i can be a variable or a constant. If t_i is a constant, it is assigned a member of D (call it d_i). If t_i is a variable, it is assigned the ith entry of s (also call it d_i). The sequence s satisfies A iff $\langle d_1, \ldots, d_n \rangle$ is a member of the set of ordered n-tuples assigned by I to the predicate symbol F.
(7) If A is of the form $\bigwedge v_k B$, where v_k is the kth variable. Then s satisfies A iff *every* sequence of members of D that differs from s in at most the kth term satisfies B.
(8) If t is a term of the form $fc_1, \ldots, c_n$, f an n-place function symbol, $c_1, \ldots, c_n$ individual constants. If f is the specific function assigned by I to f and $c_1, \ldots, c_n$ are assigned values $d_1, \ldots, d_n$ from D, then the object d assigned by I to t is the object in D that is the value of the function f for arguments $d_1, \ldots, d_n$ (i.e. $d = f(d_1, \ldots, d_n)$).
(9) If t is a term of the form $fv_1, \ldots, v_n$, f an n-place function symbol, $v_1, \ldots, v_n$ individual variables. If f is the specific function assigned by I to f and $v_1, \ldots, v_n$ are assigned values $d_1, \ldots, d_n$ from corresponding entries in s, then the object d assigned by I to t is the object in D that is the value of the function f for arguments $d_1, \ldots, d_n$ (i.e. $d = f(d_1, \ldots, d_n)$).
(10) If t is a term of the form $ft_1, \ldots, t_n$, f an n-place function symbol, where $t_1, \ldots, t_n$ are terms with function symbols followed by constants or variables. If $d_1, \ldots, d_n$ of D are assigned to $t_1, \ldots, t_n$ respectively, then the object d assigned by I to t is the object in D that is the value of the function f for arguments $d_1, \ldots, d_n$ (i.e. $d = f(d_1, \ldots, d_n)$).
(11) If t is a term of the form $ft_1, \ldots, t_n$, f an n-place function symbol, where $t_1, \ldots, t_n$ are arbitrary terms (could have terms with function symbols). If $d_1, \ldots, d_n$ of D are assigned to $t_1, \ldots, t_n$ respectively, then the object d assigned by I to t is the object in D that is the value of the function f for arguments $d_1, \ldots, d_n$ (i.e. $d = f(d_1, \ldots, d_n)$).

Definition: The function $\star$, with terms of Q (or Q^+) and values of D is defined as
(1) If t is a constant, then $t \star s$ is the member of D assigned to t by I.
(2) If t is the kth variable, then $t \star s$ is the kth entry of s.
(3) If t is of the form $ft_1, \ldots, t_n$, f an n-place function symbol, where $t_1, \ldots, t_n$ are terms, then if f is the function assigned by I to f, $t \star s = f(t_1 \star s, t_2 \star s, \ldots, t_n \star s)$.

A more concise definition of *satisfaction*:

(1) If A is a propositional truth symbol, then s satisfies A iff I assigns the truth value of T to A.

(2) If A is an atomic wff of the form $Ft_1, \ldots, t_n$, where F is an n-place predicate symbol and $t_1, \ldots, t_n$ are terms, then s satisfies A iff $\langle t_1 \star s, t_2 \star s, \ldots, t_n \star s \rangle$ is a member of the set of ordered n-tuples assigned by I to F.

(3) If A is of the $\sim B$, then s satisfies A iff s does not satisfy B.

(4) If A is of the form $B \supset C$, then s satisfies A iff either s does not satisfy B or s does satisfy C.

(5) If A is of the form $\bigwedge v_k B$, where v_k is the kth variable. Then s satisfies A iff *every* sequence of members of D that differs from s in at most the kth term satisfies B.

Definition: A formula A of Q is **satisfiable** iff there is some interpretation I of Q for which A is satisfied.
Definition: A set Γ of formulas of Q is **simultaneously satisfiable** iff for some interpretation I of Q some sequence s satisfies every member of Γ.
Definition: A wff A of Q is **true for a given interpretation I of Q** iff every denumerable sequence of members of the domain satisfies A.
Definition: A wff A of Q is **false for a given interpretation I of Q** iff no denumerable sequence of members of the domain satisfies A.
Definition: An interpretation I of Q is a **model of a formula A of Q** iff A is true for I.
Definition: An interpretation I of Q is a **model of set Γ of formulas of Q** iff every formula in Γ is true for I.
Definition: A **formal system has a model** iff the set of all its theorems has a model.
Definition: (**$\models_{\mathbf{Q}}$ A**) A formula A of Q is a **logically valid formula of Q** iff A is true for every interpretation of Q.
Definition: (**A $\models_{\mathbf{Q}}$ B**) A formula B of Q is a**semantic consequence of a formula A of Q** iff for every interpretation of Q every sequence that satisfies A also satisfies B.
Definition: (**$\Gamma \models_{\mathbf{Q}}$ B**) A formula A of Q is a **semantic consequence of a set of formulas Γ of Q** iff for every interpretation of Q every sequence that satisfies every member of Γ also satisfies B.
Definition: A formula of Q is **k-valid** iff it is true for every interpretation of Q that has a domain of exactly k members.

39.1 $\{\} \models_Q A$ iff $\models_Q A$
40.1 If A is logically valid, then $\sim A$ is not satisfiable.
40.2 Modus Ponens for $\supset$ preserves satisfaction-by-s. I.e. if a sequence s satisfies A and $A \supset B$, then it also satisfies B.
40.3 Modus Ponens for $\supset$ preserves truth-for-I. I.e. if A and $A \supset B$ are both true for an interpretation I, then B is also true for I.
40.4 Modus Ponens for $\supset$ preserves logical validity. I.e. if A and $A \supset B$ are both logically valid, then so is B. In other words, If $\models_Q A$ and $\models_Q A \supset B$, then $\models_Q B$.
40.5 A is false for a given interpretation I iff $\sim A$ is true for I; and A is true for I iff $\sim A$ is false for I.
40.6 A is true for I iff $\bigwedge vA$ is true for I for any arbitrary variable v.
40.7 A is true for I iff any arbitrary closure of A is true for I.
40.8 A is logically valid iff A^c is.
40.9 $\bigvee vA$ is satisfiable for an interpretation I iff A is satisfiable for the same interpretation.

Definition: If A is a tautology of P whose only propositional symbols are $P_1, \ldots, P_n$, then the result of substituting wffs $Q_1, \ldots, Q_n$ for $P_1, \ldots, P_n$ respectively in A is an **instance of a tautological schema of *Q***.

40.10 Every instance of a tautological schema of Q (or Q^+) is logically valid.
40.11 $\bigwedge v_k(A \supset B) \supset (\bigwedge v_k A \supset \bigwedge v_k B)$ is logically valid for arbitrary wffs A and B and an arbitrary variable v_k.

Definition: Let I be an interpretation, D its domain, s a denumerable sequence of members of D. Then $\mathbf{s(d/k)}$ is the sequence that results from replacing the kth term in the sequence s by the object d.

40.12 Let I be an interpretation with domain D. Let A be an arbitrary wff. Let s and s' be two sequences such that, for each free variable v in A, if v is the kth variable in the fixed enumeration of the variables, then s and s' have the same member of D for their kth terms. Then s satisfies A iff s' does.
40.13 If v_k does not occur free in an arbitrary wff A, then $A \supset \bigwedge v_k A$ is logically valid.

40.14 Let t and u be terms. Let t' be the result of replacing each occurrence of v_k in t by u. Let se be a sequence, and let $u \star s = d$ (i.e. the member of D assigned by I to u for the sequence s is d). Let $s' = s(d/k)$ (i.e. s' is the sequence that results from substituting d for the kth term of s). Then $t' \star s = t \star s'$ (i.e. the member of D assigned by I to t' for the sequence s is the same as the member of D assigned by I to t for the sequence s'.
40.15 Let A be a wff, v_k a variable, t a term that is free for v_k in A. Let s be a sequence, and let s' be the sequence that results from replacing the kth term of s by $t \star s$ (i.e. the member of D assigned by I to the term t for the sequence s: $s' = s(t \star s/k)$. Then s satisfies At/v_k iff s' satisfies A.
40.16 $\bigwedge v_k A \supset At/v_k$ is logically valid if t is free for v_k in A.
40.17 If A is a closed wff, then exactly one of A and $\sim A$ is true for I and exactly one is false for I.
40.18 If A and B are closed wffs, then $A \supset B$ is true for I iff A is false for I or B is true for I.
40.19 If A and B are closed wffs, then $A \supset B$ is false for I iff A is true for I and B is false for I.
40.20 If a formula A with exactly one free variable v_k is true for I, then each formula that results from substituting a closed term for the free occurrences of the variable is true for I.
40.21 Let I be an interpretation with domain D. Let A be a wff with exactly one free variable, v_k. If each member of D is assigned by I to some closed term or other, and At/v_k is true for I for each closed term t, then $\bigwedge v_k A$ is true for I.
40.22 There is an effective method for telling, given an arbitrary formula A of Q and an arbitrary positive integer k, whether or not A is k-valid.

The Deductive Apparatus for *Q* – The Formal System *QS*:

Axiom Schemas of the formal system QS
[**QS 1**] $A \supset (B \supset A)$
[**QS 2**] $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
[**QS 3**] $(\sim A \supset \sim B) \supset (B \supset A)$
[**QS 4**] $(\bigwedge vA \supset At/v)$ (if t is free for v in A)
[**QS 5**] $(A \supset \bigwedge vA)$ (if v does not occur free in A)
[**QS 6**] $(\bigwedge v(A \supset B) \supset (\bigwedge vA \supset \bigwedge vB))$
[**QS 7**] If A is an axiom, then $\bigwedge vA$ is an axiom.

Rule of Inference of (QS)
If A and B are wffs of Q, then B is an **immediate consequence** of A and $A \supset B$.

Definition: A **proof in *QS*** is a finite string of formulas of Q each one of which is either an axiom of QS or an immediate consequence by the Rule of Inference of QS of two formulas preceding it in the string.

Definition: ($\vdash_{\mathbf{QS}} \mathbf{A}$) A formula A is a theorem of QS iff there is some proof in QS whose last formula is A.

Definition: A string of formulas is a **derivation** in QS of a wff A from a set Γ of wffs of Q iff
(1) it is a finite (but not empty) string of formulas of Q
(2) the last formula in the string is A
(3) each formula of the string is one of the following three
(i) an axiom of QS
(ii) an immediate consequence by the Rule of Inference of QS of two formulas preceding it in the string
(iii) an element of the set Γ

Definition: ($\mathbf{\Gamma} \vdash_{\mathbf{QS}} \mathbf{A}$) A formula A is a **syntactic consequence** in QS of a set Γ of formulas of Q iff there is a derivation in QS of A from the set Γ.

Definition: A set Γ if formulas of Q is a **proof-theoretically consistent** set of QS iff for no formula A of Q that we have both $\vdash_{\mathbf{QS}} \mathbf{A}$ and $\vdash_{\mathbf{QS}} \sim \mathbf{A}$.

Definition: A set Γ if formulas of Q is a **proof-theoretically inconsistent** set of QS iff there exists a formula A of Q such that $\vdash_{\mathbf{QS}} \mathbf{A}$ and $\vdash_{\mathbf{QS}} \sim \mathbf{A}$.

42.1 QS is consistent.

43.1 The Deductive Theorem holds for QS
43.2 If $\Gamma \vdash QSA \supset B$, then $\{\Gamma, A\} \vdash_{QS} B$.
43.3 If Δ is a set of closed wffs, then if $\Delta \vdash_{QS} A$ then $\Delta \vdash_{QS} \bigwedge vA$.
43.4 If A is an instance of a tautological schema of Q, then $\vdash_{QS} A$.
43.5 If $\vdash_{QS} A$ then $\models_Q A$. (Every theorem of QS is logically valid)
43.6 If $\Gamma \vdash_{QS} A$, then there is a finite subset Δ of Γ such that $\Delta \vdash_{QS} A$.
43.7 If $\Gamma \vdash_{QS} A$, then $\Gamma \models_Q A$.
43.8 If $A \vdash_{QS} B$ then $\models_Q A \supset B$.
43.9 If $\Gamma \cup \{\sim A\}$ is an inconsistent set of QS, then $\Gamma \vdash_{QS} A$.