# Math 2283 - Honors Logic

## Midterm - 2018.10.22

## Solutions

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**Potentially Useful Information**:

*Tautological Sentences*:

(1) $\sim (p \vee q) \longleftrightarrow \sim p \wedge \sim q$
(2) $\sim (p \wedge q) \longleftrightarrow \sim p \vee \sim q$
(3) $p \vee q \longleftrightarrow \sim (\sim p \wedge \sim q)$
(4) $p \wedge q \longleftrightarrow \sim (\sim p \vee \sim q)$
(5) $(p \rightarrow q) \longleftrightarrow \sim p \vee q$
(6) $(p \wedge q) \longrightarrow p$
(7) $(p \wedge q) \longrightarrow q$
(8) $(p \wedge q) \longleftrightarrow (q \wedge p)$
(9) $p \wedge T \longleftrightarrow p$ (Here $T$ is any true sentence)
(10) $p \vee F \longleftrightarrow p$ (Here $F$ is any false sentence)
(11) $\forall x, y, z\, P(x, y, z) \longrightarrow \forall x, y\, P(x, y, x)$

*Class Properties*:

$x \in K' \overset{def}{\longleftrightarrow} \sim x \in K$
$x \in K \cup L \overset{def}{\longleftrightarrow} x \in K \vee x \in L$
$x \in K \cap L \overset{def}{\longleftrightarrow} x \in K \wedge x \in L$

*Relation Properties*:

$R$ is symmetric $\overset{def}{\longleftrightarrow} \forall\, x, y \in K\ xRy \rightarrow yRx$
$R$ is reflexive $\overset{def}{\longleftrightarrow} \forall\, x \in K\ xRx$
$R$ is transitive $\overset{def}{\longleftrightarrow} \forall\, x \in K\ xRy \wedge yRz \longrightarrow xRz$

0. State the following as clearly as possible:

(a) The Rule of Detachment

If two sentences are accepted as true, of which one has the form of an implication while the other is the antecedent of this implication, then that sentence may also be recognized as true, which forms the consequent of the implication. (We *detach* thus, so to speak, the antecedent from the whole implication.).

(b) The Rule of Substitution

If a sentence of a universal character, that has already been accepted as true, contains sentential variables, and if these variables are replaced by other sentential variables or by sentential functions or sentences – always substituting equal expressions for equal variables throughout –, then the sentence obtained in this way may be recognized as true.

(c) The Rule of And, Joining Together

If any two sentences are accepted as true, then their conjunction may be recognized as true.

1. Construct a sentential function using the standard connectives and the sentential variables $p$, $q$, and $r$ whose truth table is given below:

There are many sentential functions and many approaches to find them, but consider the two false rows. The first happens when $p$ is true, $q$ is not true and $r$ is true. Thus, symbolically, this means that $p \wedge \sim q \wedge r$ is true for this row, which means that $\sim (p \wedge \sim q \wedge r)$ is false. Similarly, for row 5, we have that $\sim p \wedge q \wedge r$ is true, thus $\sim (\sim p \wedge q \wedge r)$ is false. The claim is the following sentence generates the truth table:

$$\sim (p \wedge \sim q \wedge r) \wedge \sim (\sim p \wedge q \wedge r)$$

| $p$ | $q$ | $r$ | $\sim (p \wedge \sim q \wedge r)$ | $\sim (\sim p \wedge q \wedge r)$ | $\sim (p \wedge \sim q \wedge r) \wedge \sim (\sim p \wedge q \wedge r)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | F | T | F |
| T | F | F | T | T | T |
| F | T | T | T | F | F |
| F | T | F | T | T | T |
| F | F | T | T | T | T |
| F | F | F | T | T | T |

2. The connective $\downarrow$ is defined by the following fundamental truth table:

| $p$ | $q$ | $p \downarrow q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

We know that the conditional connective $\rightarrow$ can be written in terms of negation, $\sim$, and disjunction, $\vee$, because the truth table for $p \rightarrow q$ is exactly the same as that of $\sim p \vee q$. In a similar fashion:

(a) express $\sim p$ using *ONLY* the $\downarrow$ connective.

The most obvious sentential function to check first is $p \downarrow p$:

| $p$ | $p \downarrow p$ |
|---|---|
| T | F |
| F | T |

And indeed, we do see that $(p \downarrow p) \leftrightarrow \sim p$.

(b) express $p \wedge q$ using *ONLY* the $\downarrow$ connective.

The idea here is to look at the truth values for both $p \downarrow p$ and $q \downarrow q$. Since we have only one connective, the last column follows immediately, which is the truth table definition of $p \wedge q$.

| $p$ | $q$ | $p \downarrow p$ | $q \downarrow q$ | $(p \downarrow p) \downarrow (q \downarrow q)$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | F |

3. A relation $R$ is said to be **Euclidean** in the class $K$ if for any $x$, $y$ and $z$ in $K$, if $xRy$ and $xRz$, then $yRz$ I.e.

$$R \text{ is } \textit{Euclidean} \overset{def}{\longleftrightarrow} \forall\, x, y, z \in K \ (xRy \wedge xRz) \to yRz.$$

Prove the following theorem of relations: If a relation $R$ is Euclidean and reflexive on a class $K$, then the relation $R$ is symmetric on the class $K$.

| | | |
|---|---|---|
| (1) | $R$ is Euclidean $\wedge\, R$ is reflexive | Assume Hypothesis |
| (2) | ($R$ is Euclidean $\wedge\, R$ is reflexive) $\longrightarrow R$ is Euclidean | Instance of $p \wedge q \to p$<br>$p : R$ is Euclidean, $q : R$ is reflexive |
| (3) | ($R$ is Euclidean $\wedge\, R$ is reflexive) $\longrightarrow R$ is reflexive | Instance of $p \wedge q \to q$<br>$p : R$ is Euclidean, $q : R$ is reflexive |
| (4) | $R$ is Euclidean | R.O.D. on (1) and (2) |
| (5) | $R$ is reflexive | R.O.D. on (1) and (3) |
| (6) | $R$ is Euclidean $\overset{def}{\longleftrightarrow} \forall\, x, y, z \in K \ (xRy \wedge xRz) \to yRz$ | Def. of $R$ Euclidean |
| (7) | $R$ is reflexive $\overset{def}{\longleftrightarrow} \forall\, x \in K \ xRx$ | Def. of $R$ reflexive |
| (8) | $\forall\, x, y, z \in K \ (xRy \wedge xRz) \to yRz$ | Subs. (6) into (4) |
| (9) | $\forall\, x \in K \ xRx$ | Subs. (7) into (5) |
| (10) | $(\forall\, x, y, z \in K \ (xRy \wedge xRz) \to yRz)$<br>$\longrightarrow (\forall\, x, y \in K \ (xRy \wedge xRx) \to yRx)$ | Inst. of $P(x, y, z) \to P(x, y, x)$,<br>$P(x, y, z) : (xRy \wedge xRz) \to yRz$ |
| (11) | $\forall\, x, y \in K \ (xRy \wedge xRx) \to yRx$ | R.O.D. on (8) and (10) |
| (12) | $xRy \wedge xRx \longleftrightarrow xRy$ | Instance of $p \wedge T \leftrightarrow p$<br>$p : xRy,\ T : (9)$ |
| (13) | $\forall\, x, y \in K \ xRy \to yRx$ | Subs. (12) into (11) |
| (14) | $R$ is symmetric $\overset{def}{\longleftrightarrow} \forall\, x, y \in K \ xRy \to yRx$ | Def. of $R$ symmetric |
| (15) | $R$ is symmetric | Subs. (14) into (13) |

□

4. Define the class operation $\Delta$ on two classes $K$ and $L$ as follows:

$$x \in K\Delta L \overset{def}{\longleftrightarrow} (x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L)$$

In a sentence, elements of the class $K\Delta L$ belong to either $K$ or $L$ but not in the intersection of $K$ and $L$. Prove the following identity: $K\Delta L = K'\Delta L'$.

| | | |
|---|---|---|
| (1) | $K\Delta L = K'\Delta L' \overset{def}{\longleftrightarrow} x \in K\Delta L \longleftrightarrow x \in K'\Delta L'$ | Inst. Def. of $K = L$ <br> $K : K\Delta L,\ L : K'\Delta L'$ |
| (2) | $x \in K\Delta L \overset{def}{\longleftrightarrow} (x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L)$ | Definition of $K\Delta L$ |
| (3) | $(x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L) \longleftrightarrow$ <br> $\sim (x \in K \wedge x \in L) \wedge (x \in K \vee x \in L)$ | Instance of $p \wedge q \leftrightarrow q \wedge p$ <br> $p : (x \in K \vee x \in L)$, $q :\sim (x \in K \wedge x \in L)$ |
| (4) | $x \in K\Delta L \longleftrightarrow \sim (x \in K \wedge x \in L) \wedge (x \in K \vee x \in L)$ | Subs (3) into (2) |
| (5) | $\sim (x \in K \wedge x \in L) \longleftrightarrow (\sim x \in K \vee \sim x \in L)$ | Instance of $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$ <br> $p : x \in K$, $q : x \in L$ |
| (6) | $x \in K\Delta L \longleftrightarrow (\sim x \in K \vee \sim x \in L) \wedge (x \in K \vee x \in L)$ | Subs (5) into (4) |
| (7) | $x \in K \vee x \in L \longleftrightarrow \sim (\sim x \in K \wedge \sim x \in L)$ | Instance of $p \vee q \leftrightarrow \sim (\sim p \wedge \sim q)$ <br> $p : x \in K$, $q : x \in L$ |
| (8) | $x \in K\Delta L \longleftrightarrow (\sim x \in K \vee \sim x \in L) \wedge \sim (\sim x \in K \wedge \sim x \in L)$ | Subs (7) into (6) |
| (9) | $x \in K' \overset{def}{\longleftrightarrow} \sim x \in K$ | Definition of $K'$ |
| (10) | $x \in L' \overset{def}{\longleftrightarrow} \sim x \in L$ | Inst. Def. of $K'$, $K : L$ |
| (11) | $x \in K\Delta L \longleftrightarrow (x \in K' \vee x \in L') \wedge \sim (x \in K' \wedge x \in L')$ | Subs (9) and (10) into (8) |
| (12) | $x \in K'\Delta L' \overset{def}{\longleftrightarrow} (x \in K' \vee x \in L') \wedge \sim (x \in K' \wedge x \in L')$ | Inst. Def. of $K\Delta L, K : K', L : L'$ |
| (13) | $K\Delta L = K'\Delta L'$ | Subs. (12) in (11) |

□

5. Simply place: true or false, into each box below.

T T T T T T T T