Math 2283 - Introduction to Logic
Final Exam

Assigned: 2019.04.24

Due: 2019.05.08 at 08:00

Instructions: Work on this by yourself, the only person you may contact in any way to discuss or ask questions about this exam is Dr. Frinkle. For each problem, be sure to show all of your work and write every step down in a clear and concise manner. Please start each problem on a new sheet. When complete, staple all sheets in order to the cover page. You do not have to attach the remaining pages containing the actual questions if you do not so desire. Remember, you have two whole weeks to work on this, your masterpiece will be graded accordingly.

Agreement: Please read the following statement and then write it at the bottom of the page before the signature line:

"*I hereby swear that all the work that appears on this written exam is completely my own, and I have not discussed any portion of this exam with any one else besides the instructor.*"

Printed Name: ____________________________________

Signature: ____

Date: ____

Part I

Definitions:

A relation R is *co-reflexive* if and only if $\forall\, x, y,\ xRy \to x = y$.

A relation R is *anti-symmetric* if and only if $\forall\, x, y,\ xRy \wedge yRx \to x = y$.

A relation R is *left unique* if and only if $\forall\, x_1, x_2, y,\ x_1Ry \wedge x_2Ry \to x_1 = x_2$.

A relation R is *right quasi-reflexive* if and only if $\forall\, x, y,\ xRy \to yRy$.

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0. (10 pts) If you look at problems 1–5 below, some of the properties you are to prove involve conditional sentences. For instance, in problem 3, you are to prove R is co-reflexive assuming R is right quasi-reflexive and left unique. The definition of co-reflexive is itself the conditional sentence $\forall\, x, y,\ xRy \to x = y$. One might be able to derive the entire definition of co-reflexive using only the assumptions of right quasi-reflexiveness and left uniqueness. However, it is easier if one assumes the hypothesis of the property of co-reflexiveness, and then simply derives $x = y$ using the properties of right quasi-reflexiveness and left uniqueness. To do this, one must argue that this is indeed a valid approach. We do this in parts (a) and (b) as follows:

(a) Construct a truth table for the following sentential function: $[(p \wedge q \wedge r) \to s] \longleftrightarrow [(p \wedge q) \to (r \to s)]$

(b) Using the result from (a), argue why the aforementioned approach to proving certain conditional theorems is valid (and can thus be used in problems 1–4 if so desired).

1. (15 pts) Prove that if a relation R is anti-symmetric, then R' is connected.

2. (15 pts) Prove that if a relation R is reflexive, then R is not asymmetric.

3. (15 pts) Prove that if a relation R is right quasi-reflexive and left unique, then R is also co-reflexive.

4. (15 pts) Prove that if a relation R is left unique and transitive, then R is also anti-symmetric.

5. (15 pts) Prove the following theorem:

$$p \to [\ (p \to q) \to q\ ]$$

using ONLY the rule of substitution and the law of detachment along with the following two theorems:

Theorem I. $[\ p \to (q \to r)] \to [q \to (p \to r)]$
Theorem II. $p \to p$

6. (15 pts) Prove the following theorem directly. (I.e. you cannot prove by the method of truth tables)

$$[(p \vee q) \wedge (p \to r)] \to (q \vee r)]$$

7. (30 pts) Define axiomatic systems (A) and (B) on a class K with relation R as follows:

Axiomatic system (A):
Axiom 1^A. The relation R is connected in the class K.
Axiom 2^A. The relation R is asymmetric in the class K.
Axiom 3^A. The relation R is transitive in the class K.

Axiomatic system (B):

Axiom 1^B**.** The relation R is connected in the class K.
Axiom 2^B**.** $(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \rightarrow \sim vRx$

Recall that Axiomatic system (A) implies that the relation R orders the class K. Prove that axiomatic systems (A) and (B) are equipollent. What does this imply about the definition of an ordering of a class K using a relation R?

8. (40 pts) A Deductive Theory Example. For this problem, you are allowed to use only the Rule of Detachment as a rule of inference (hence no Rule of Substitution). You are also allowed to use any tautological sentence, however if you use a non-standard logical law, please explicitly state the form of said law. Suggested logical laws are given for specific theorems. The universe of discourse shall be defined to be an unspecified collection of classes, denoted by capital letters such as K, L, and M. The primitive terms will be the relations "$\subseteq$" (subclass of) and "$\asymp$" (disjoint). Lastly, we will allow for the use of the following five axioms:

Axiom I. $K \subseteq K$
Axiom II. $(K \subseteq L \wedge L \subseteq M) \rightarrow K \subseteq M$
Axiom III. $M \asymp N \rightarrow N \asymp M$
Axiom IV. $(K \subseteq L \wedge L \asymp M) \rightarrow K \asymp M$
Axiom V. $K \asymp L \rightarrow \sim K \subseteq L$

Prove the following theorems under the restrictions described. You may *not* assume the hypothesis of any conditional theorem to prove said theorem. Once a theorem has been proven, it may be used in the proofs of subsequent theorems.

Theorem 1. $M \asymp N \leftrightarrow N \asymp M$ *Hint*: Use the law $(p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]$
Theorem 2. $(K \subseteq L \wedge M \asymp L) \rightarrow K \asymp M$ *Hint*: Use the law $(p \leftrightarrow q) \rightarrow \{[(r \wedge p) \rightarrow s] \rightarrow [(r \wedge q) \rightarrow s]\}$
Theorem 3. $L \asymp K \rightarrow \sim K \subseteq L$ *Hint*: Use the law $(p \leftrightarrow q) \rightarrow \{(q \rightarrow r) \rightarrow (p \rightarrow r)\}$
Theorem 4. $K \asymp L \rightarrow (\sim K \subseteq L \wedge \sim L \subseteq K)$ *Hint*: Use the law $(p \rightarrow q) \rightarrow \{(p \rightarrow r) \rightarrow [p \rightarrow (q \wedge r)]\}$
Theorem 5. $K \subseteq L \rightarrow \sim K \asymp L$ *Hint*: Use the law $(p \rightarrow \sim q) \rightarrow (q \rightarrow \sim p)$
Theorem 6. $\sim K \asymp K$ *Hint*: At some point, Theorem 5 may come in handy.
Theorem 7. $(K \subseteq L \wedge L \subseteq M) \rightarrow \sim K \asymp M$ *Hint*: Use the law $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$
Theorem 8. $(K \subseteq L \wedge L \asymp M) \rightarrow \sim K \subseteq M$ *Hint*: Similar to Theorem 7.

Part II

Propositional System Ξ:

Symbols

The system uses lower case letters such as p, q, and r, to represent propositional symbols.
The symbol $\vee$ is the disjunctive connective: $p \vee q$ means 'p or q'.
$\overline{}$ is the symbol for negation: $\overline{p}$ is the negation of sentence p, while $\overline{p \vee q}$ is the negation of the sentence $p \vee q$.

Well Formed Formulas (WFFs)

(1) Any propositional symbol is a wff.
(2) If A is a wff, then $\overline{A}$ is a wff.
(3) If A and B are wffs, then $A \vee B$ is a wff.
(4) Nothing else is a wff.

Examples:

(1) p is a wff.
(2) $\overline{p} \vee q$ is a wff.
(3) $\overline{\overline{p \vee q} \vee r}$ is a wff.
(4) $\overline{\overline{p \vee q}\vee}r$ is not a wff.

Definition: A wff A is a *primitive disjunction* iff A is of the form $B_1 \vee B_2 \vee \ldots \vee B_n$ where each B_i is either a propositional symbol or a propositional symbol with a bar over it (i.e. each B_i is of the form p or $\overline{p}$).

Examples:

(1) $p \vee \overline{q} \vee r$ is a primitive disjunction.
(2) $\overline{p \vee \overline{q}}$ is not a primitive disjunction.
(3) $\overline{\overline{p}} \vee q$ is not a primitive disjunction.
(4) $\overline{p} \vee \overline{q} \vee q \vee p$ is a primitive disjunction.

Definition: Every wff A is a *disjunctive part* of itself, and if A is a *disjunctive part* of $B \vee C$, then B is a *disjunctive part* of A and so is C.

Examples:

(1) p is a disjunctive part of p.
(2) $\overline{p}$ is a disjunctive part of $q \vee \overline{p} \vee p$.
(3) $\overline{p} \vee p$ is a disjunctive part of $q \vee \overline{p} \vee p$.
(4) $q \vee p$ is a not disjunctive part of $q \vee \overline{p} \vee p$.

Notation: $D(A)$ is a wff of which A is a disjunctive part, and $D(B)$ is the result of replacing *one occurrence* of the disjunctive part A in $D(A)$ by B.

Examples:

(1) If $D(p)$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $q \vee \overline{r} \vee \overline{p} \vee q$.
(2) If $D(q)$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $p \vee \overline{p} \vee q \vee \overline{r}$.
(3) If $D(\overline{p})$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $p \vee q \vee \overline{r} \vee q$.
(4) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is not $\overline{r} \vee \overline{p} \vee \overline{r} \vee r$.
(5) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is $p \vee \overline{p} \vee \overline{r} \vee r$.
(6) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is $\overline{r} \vee \overline{p} \vee p \vee r$.

Axioms of System Ξ:

There are an infinite number of axioms for System Ξ. A wff A is an axiom iff A is a primitive disjunction and for some propositional symbol B, both B and $\overline{B}$ are disjunctive parts of A.

Examples:

(1) $q \vee \overline{r} \vee \overline{q} \vee p$ is an axiom.
(2) $q \vee \overline{r} \vee \overline{q \vee \overline{r}}$ is not an axiom.
(3) $\overline{\overline{q}} \vee \overline{q}$ is not an axiom.
(4) $q \vee \overline{r} \vee \overline{q} \vee \overline{q \vee \overline{r}}$ is not an axiom.
(5) $\overline{q} \vee r \vee p \vee \overline{q} \vee \overline{r}$ is an axiom.

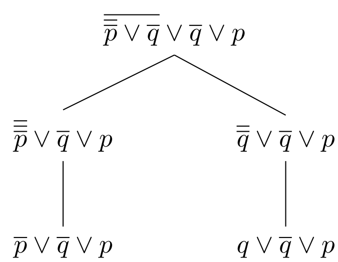
Rules of Inference for System Ξ

Rule I: $D(\overline{\overline{A}})$ is an immediate consequence of $D(A)$.
Rule II: $D(\overline{A \vee B})$ is an immediate consequence of $D(\overline{A})$ and $D(\overline{B})$

Examples:

(1) $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
(2) $p \vee \overline{\overline{\overline{r}}} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
(3) $\overline{\overline{p} \vee q} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ and $\overline{q} \vee \overline{r} \vee \overline{p} \vee q$ by Rule II.
(4) $q \vee \overline{\overline{p} \vee q} \vee \overline{p} \vee r$ is an immediate consequence of $q \vee \overline{\overline{p}} \vee \overline{p} \vee r$ and $q \vee \overline{q} \vee \overline{p} \vee r$ by Rule II.

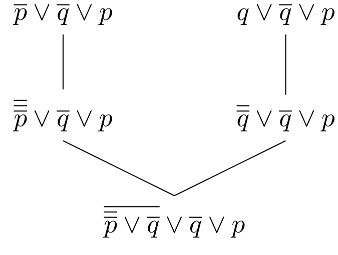
There exists a method of proof in System Ξ to derive any tautological sentence which can be aided by the construction of a tree. An example of this method is given in the two trees below for the sentence $\overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p$. Note that the trees are simply reflections of each other, one growing downwards, the second growing upwards.



The first tree (which grows downwards) is used to dissect the original sentence using Rules I and II and the axioms given. Each branch of the tree terminates at an axiom. Notice, in the lowest branch of the above tree, the two sentences $\overline{p} \vee \overline{q} \vee p$ and $q \vee \overline{q} \vee p$ are axioms, as the first contains both p and $\overline{p}$, while the second contains both q and $\overline{q}$.

To build the tree, work from left to right and pick the leftmost portion of the wff which is not in primitive disjunctive form. For instance, $\overline{\overline{\overline{p}} \vee \overline{q}}$ is the only portion of the original sentence not in primitive disjunctive form. Only an application of Rule II would yield $\overline{\overline{\overline{p}} \vee \overline{q}}$ as part of a wff, so we simply determine the two wffs so that when Rule II is applied to them, we get $\overline{\overline{\overline{p}} \vee \overline{q}}$. The two wffs required are $\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$ and $\overline{\overline{q}} \vee \overline{q} \vee p$.

At each stage, we check to see if the wffs used are primitive disjunctions. In this example, they are not, but it can be easily seen that Rule I can be used on $\overline{p} \vee \overline{q} \vee p$ to arrive at $\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$, and can also be used on $q \vee \overline{q} \vee p$ to arrive at $\overline{\overline{q}} \vee \overline{q} \vee p$. Now every branch in the tree above ends in a primitive disjunction. Furthermore, they are axioms and thus we can build a proof to generate the sentence in question from axioms using only Rules I and II. To make it easier to see this, we invert the tree as below and build the proof.



The following is the proof generated by the tree given above. Note that only axioms, and Rules I and II are used. This is the approach to be taken for any wff in this system which you believe is a tautology.

(1)	$q \vee \overline{q} \vee p$	Axiom
(2)	$\overline{p} \vee \overline{q} \vee p$	Axiom
(3)	$\overline{\overline{q}} \vee \overline{q} \vee p$	Rule I applied to (1)
(4)	$\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$	Rule I applied to (2)
(5)	$\overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p$	Rule II applied to (4) and (3)

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1. (15 pts) Explain why the conditions stated to be an axiom of System Ξ imply that every axiom is a tautology.

2. (20 pts) Construct arguments that Rules I and II of System Ξ indeed preserve the notion of a tautology. I.e. if Rule I is applied to a tautological sentence A, then the result is also a tautological sentence. Similarly, if Rule II is applied to two tautological sentences A and B, the result is also a tautology. You do not need a rigorous proof, but your argument should be convincing.

3. (10 pts) Is System Ξ simply and/or absolutely consistent? Explain your answer thoroughly.

4. (10 pts) Is System Ξ semantically complete with respect to the axioms and two rules of inference previously specified? Explain your answer thoroughly.

5. (10 pts) Express the logical sentence $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$ as a valid sentence of System Ξ. I.e. write it in terms of negations and disjunctions in such a way that it is a wff of System Ξ.

6. (10 pts) Create a tree for the wff you constructed in the previous problem.

7. (10 pts) Using the tree from the previous problem, construct a proof using only axioms, and Rules I and II, that shows your wff is a tautology.

8. (30 pts) Starting with the sentence $(p \leftrightarrow q) \rightarrow \{[(r \wedge p) \rightarrow s] \rightarrow [(r \wedge q) \rightarrow s]\}$, repeat the work done in problems 5, 6, and 7. I.e. convert the given logical law to a wff of System Ξ and prove it is a tautology in the same manner as you did for $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$.