Math 2283 - Honors Logic

Midterm - 2019.03.27

Solutions

Potentially Useful Information:

Tautological Sentences:

(1) $\sim (p \vee q) \longleftrightarrow \sim p \wedge \sim q$
(2) $\sim (p \wedge q) \longleftrightarrow \sim p \vee \sim q$
(3) $p \vee q \longleftrightarrow \sim (\sim p \wedge \sim q)$
(4) $p \wedge q \longleftrightarrow \sim (\sim p \vee \sim q)$
(5) $(p \to q) \longleftrightarrow \sim p \vee q$
(6) $(p \vee \sim p) \longleftrightarrow T$
(7) $(\sim p \vee p) \longleftrightarrow T$
(8a) $(p \wedge q) \longleftrightarrow (q \wedge p)$
(8b) $(p \vee q) \longleftrightarrow (q \vee p)$
(9a) $p \wedge T \longleftrightarrow p$ (Here T is any true sentence)
(9b) $T \wedge p \longleftrightarrow p$ (Here T is any true sentence)
(10) $p \vee F \longleftrightarrow p$ (Here F is any false sentence)
(11a) $p \wedge (q \vee r) \longleftrightarrow (p \wedge q) \vee (p \wedge r)$
(11b) $(p \vee q) \wedge r \longleftrightarrow (p \wedge r) \vee (q \wedge r)$
(12a) $p \vee (q \wedge r) \longleftrightarrow (p \vee q) \wedge (p \vee r)$
(12b) $(p \wedge q) \vee r \longleftrightarrow (p \vee r) \wedge (q \vee r)$

Class Properties:

$x \in K' \overset{def}{\longleftrightarrow} \sim x \in K$
$x \in K \cup L \overset{def}{\longleftrightarrow} x \in K \vee x \in L$
$x \in K \cap L \overset{def}{\longleftrightarrow} x \in K \wedge x \in L$

Relation Properties:

R is symmetric $\overset{def}{\longleftrightarrow} \forall x, y \in K \ xRy \to yRx$
R is reflexive $\overset{def}{\longleftrightarrow} \forall x \in K \ xRx$
R is transitive $\overset{def}{\longleftrightarrow} \forall x \in K \ xRy \wedge yRz \longrightarrow xRz$

0. State the following as clearly as possible:

(a) The Rule of Detachment

If two sentences are accepted as true, of which one has the form of an implication while the other is the antecedent of this implication, then that sentence may also be recognized as true, which forms the consequent of the implication.

(b) The Rule of Substitution

If a sentence of a universal character, that has already been accepted as true, contains sentential variables, and if these variables are replaced by other sentential variables or by sentential functions or sentences – always substituting equal expressions for equal variables throughout –, then the sentence obtained in this way may be recognized as true.

(c) The Rule of And, Joining Together

If any two sentences are accepted as true, then their conjunction may be recognized as true.

1. We know that the disjunction of two sentences can be written in terms of negation and conditional connectives, i.e. $(p \vee q) \longleftrightarrow (\sim p \rightarrow q)$. In a similar fashion, express (a) $p \wedge q$ and (b) $p \longleftrightarrow q$ using only $\sim$, $\rightarrow$ and parentheses (,).

We start with $p \wedge q$, which we know is equivalent to $\sim (\sim p \vee \sim q)$ by DeMorgan's Law. But this contains a disjunction, which we can rewrite in terms of $\sim$ and $\rightarrow$. Therefore, we have that $p \wedge q$ is equivalent to $\sim (p \rightarrow \sim q)$, after using the identity $\sim (\sim p) \longleftrightarrow q$.

After this, we know that $p \longleftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, which we can rewrite using everything we have done so far. In steps, first we rewrite the conjunction using DeMorgan's Law: $\sim (\sim (p \rightarrow q) \vee \sim (q \rightarrow p))$. Then we rewrite the disjunction as done previously to get $\sim ((p \rightarrow q) \rightarrow \sim (q \rightarrow p))$.

2. A relation R is said to be **Co-Reflexive** in the class K if for any x and y in K, if xRy, then $x = y$ I.e.

$$R \text{ is } \textit{Co-Reflexive} \overset{def}{\longleftrightarrow} \forall x, y \in K \ xRy \rightarrow x = y.$$

A relation R is said to be **Antisymmetric** in the class K if for any x and y in K, if xRy and yRx, then $x = y$. In logical definition form:

$$R \text{ is } \textit{Antisymmetric} \overset{def}{\longleftrightarrow} \forall x, y \in K \ xRy \wedge yRx \rightarrow x = y.$$

Prove the following theorem of relations:

Theorem: If a relation R is symmetric and antisymmetric on a class K, then the relation R is co-reflexive on the class K.

Hint: Assume the hypothesis of the definition of R being co-reflexive, namely: xRy, and then, using the definitions of R being symmetric and antisymmetric, derive the conclusion to the definition of R being co-reflexive: $x = y$.

(1)	xRy	Assume Hypothesis
(2)	$xRy \rightarrow yRx$	Def. of R symmetric
(3)	yRx	R.O.D. on (1) and (2)
(4)	$xRy \wedge yRx$	R.O.A.J.T. on (3) and (1)
(5)	$xRy \wedge yRx \rightarrow x = y$	Def. of R antisymmetric
(6)	$x = y$	R.O.D. on (4) and (5)

3. Express the properties of (a) co-reflexivity and (b) antisymmetry of a relation R using relations. As a reminder, the property of asymmetry for a relation R can be expressed by $R \subseteq (\breve{R})'$. Also, I was the relation of being *identical*, and D for *different*.

(a) Co-reflxivity: $R \subseteq I$
(b) Antisymmetric: $R \cap \breve{R} \rightarrow I$

4. Define the class operation Δ on two classes K and L as follows:

$$x \in K \Delta L \overset{def}{\longleftrightarrow} (x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L)$$

In a sentence, elements of the class $K \Delta L$ belong to either K or L but not in the intersection of K and L.

Furthermore, define the class operation $-$ on two classes K and L as follows:

$$x \in K - L \overset{def}{\longleftrightarrow} x \in K \wedge \sim x \in L$$

In a sentence, elements of the class $K - L$ belong to K and do not belong to L.

Prove the following class based theorem:

Theorem: $K\Delta L = (K-L)\cup(L-K)$

(1)	$x \in (K-L)\cup(L-K) \overset{def}{\longleftrightarrow} x \in K-L \vee x \in L-K$	Inst. def. of $K\cup L$ $K: K-L,\ L: L-K$
(2)	$x \in (K-L) \overset{def}{\longleftrightarrow} x \in K \wedge \sim x \in L$	Definition of $K-L$
(3)	$x \in (L-K) \overset{def}{\longleftrightarrow} x \in L \wedge \sim x \in K$	Inst. def. of $K-L, K: L, L: K$
(4)	$x \in (K-L)\cup(L-K) \longleftrightarrow$ $(x\in K\wedge \sim x\in L)\vee(x\in L\wedge \sim x\in K)$	Subs (2),(3) into (1)
(5)	$(x\in K\wedge \sim x\in L)\vee(x\in L\wedge \sim x\in K) \longleftrightarrow$ $[(x\in K\wedge \sim x\in L)\vee x\in L]\wedge[(x\in K\wedge \sim x\in L)\vee \sim x\in K]$	Inst. of Law 12a, $p: x\in K\wedge \sim x\in L$ $q: x\in L,\ r: \sim x\in K,$
(6)	$x \in (K-L)\cup(L-K) \longleftrightarrow$ $[(x\in K\wedge \sim x\in L)\vee x\in L]\wedge[(x\in K\wedge \sim x\in L)\vee \sim x\in K]$	Subs (5) into (4)
(7)	$(x\in K\wedge \sim x\in L)\vee x\in L \longleftrightarrow$ $(x\in K\vee x\in L)\wedge(\sim x\in L\vee x\in L)$	Inst. of Law 12b, $p: x\in K,$ $q: \sim x\in L, r: x\in L$
(8)	$(x\in K\wedge \sim x\in L)\vee \sim x\in K \longleftrightarrow$ $(x\in K\vee \sim x\in K)\wedge(\sim x\in L\vee \sim x\in K)$	Inst. of Law 12b, $p: x\in K,$ $q: \sim x\in L, r: \sim x\in K$
(9)	$\sim x\in L\vee x\in L \longleftrightarrow T$	Inst. of Law 7, $p: x\in L$
(10)	$x\in K\vee \sim x\in K \longleftrightarrow T$	Inst. of Law 6, $p: x\in K$
(11)	$(x\in K\vee x\in L)\wedge(\sim x\in L\vee x\in L) \longleftrightarrow$ $(x\in K\vee x\in L)\wedge T$	Subs. (9) into (7)
(12)	$(x\in K\vee \sim x\in K)\wedge(\sim x\in L\vee \sim x\in K) \longleftrightarrow$ $T\wedge(\sim x\in L\vee \sim x\in K)$	Subs (10) into (8)
(13)	$(x\in K\vee x\in L)\wedge T \longleftrightarrow x\in K\vee x\in L$	Inst. of Law 9a, $p: x\in K\vee x\in L$
(14)	$T\wedge(\sim x\in L\vee \sim x\in K) \longleftrightarrow \sim x\in L\vee \sim x\in K$	Inst. of Law 9b, $p: \sim x\in L\vee \sim x\in K$
(15)	$(x\in K\vee x\in L)\wedge(\sim x\in L\vee x\in L) \longleftrightarrow$ $(x\in K\vee x\in L)$	Subs (13) into (11)
(16)	$(x\in K\vee \sim x\in K)\wedge(\sim x\in L\vee \sim x\in K) \longleftrightarrow$ $(\sim x\in L\vee \sim x\in K)$	Subs (14) into (12)
(17)	$(x\in K\wedge \sim x\in L)\vee x\in L \longleftrightarrow$ $(x\in K\vee x\in L)$	Subs (15) into (7)
(18)	$(x\in K\wedge \sim x\in L)\vee \sim x\in K \longleftrightarrow$ $(\sim x\in L\vee \sim x\in K)$	Subs (16) into (8)

(19) $x \in (K - L) \cup (L - K) \longleftrightarrow$ Subs (17), (18) into (4)

$(x \in K \vee x \in L) \wedge (\sim x \in L \vee \sim x \in K)$

(20) $(\sim x \in L \vee \sim x \in K) \longleftrightarrow (\sim x \in K \vee \sim x \in L)$ Inst. of Law 8b, $p :\sim x \in L, q :\sim x \in K$

(21) $x \in (K - L) \cup (L - K) \longleftrightarrow$ Subs (20) into (19)

$(x \in K \vee x \in L) \wedge (\sim x \in K \vee \sim x \in L)$

(22) $\sim (x \in K \wedge x \in L) \longleftrightarrow (\sim x \in K \vee \sim x \in L)$ Inst. of Law 2, $p : x \in K, q : x \in L$

(23) $x \in (K - L) \cup (L - K) \longleftrightarrow$ Subs (22) into (21)

$(x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L)$

(24) $x \in K \Delta L \overset{def}{\longleftrightarrow} (x \in K \vee x \in L) \wedge \sim (x \in K \wedge x \in L)$ Def. of $K \Delta L$

(25) $x \in (K - L) \cup (L - K) \longleftrightarrow x \in K \Delta L$ Subs. (24) into (23)

(26) $K \Delta L = (K - L) \cup (L - K) \overset{def}{\longleftrightarrow}$ Inst. def. of $K = L$,

$x \in K \Delta L \longleftrightarrow x \in (K - L) \cup (L - K)$ $K : K \Delta L, L : (K - L) \cup (L - K)$

(27) $K \Delta L = (K - L) \cup (L - K)$ Subs. (26) into (25)