Math 3283 - Foundations of Mathematics

Exam 1 - 2019.02.11

Solutions

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1. Construct a the truth table for the sentence $((p \vee q) \to r) \leftrightarrow ((p \to r) \wedge (q \to r))$.

p	q	r	$p \vee q$	(A) $(p \vee q) \to r$	$p \to r$	$q \to r$	(B) $(p \to r) \wedge (q \to r)$	$A \leftrightarrow B$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

2. Express the negation of the sentence $\forall x, y\, \exists z\ (P(x,y) \to Q(z)) \vee (P(x,y) \to R(z))$. Do not use $\to$ in your final answer.

The quantifiers are easy to negate, and we have a disjunction, which negated becomes a conjunction. Then using the fact that $p \to q \Leftrightarrow \sim p \vee q$ and thus $\sim p \to q \Leftrightarrow p \wedge \sim q$, we have

$$\begin{aligned}\sim \forall x, y\, \exists z\ (P(x,y) \to Q(z)) \vee (P(x,y) \to R(z)) &\Leftrightarrow \exists x, y\, \forall z\ \ \sim (P(x,y) \to Q(z)) \wedge \sim (P(x,y) \to R(z)) \\ &\Leftrightarrow \exists x, y\, \forall z\ (P(x,y) \wedge \sim Q(z)) \wedge (P(x,y) \wedge \sim R(z))\end{aligned}$$

3. Define $P(x,y)$ to be the predicate function "Person x is *as tall as, or taller* than person y". Determine whether each of the following quantified sentences is true or false. You may assume the domain of discourse is the set of people in this classroom.

(a) $\exists x, y\ P(x,y)$

Yes there are two people for which one is taller than another.

(b) $\exists x\, \forall y\ P(x,y)$

Yes, there is a tallest person (who is at least as tall as themselves.

(c) $\exists y\, \forall x\ P(x,y)$

Yes, there is a shortest person (who is as short as themselves).

(d) $\forall x, y\ P(x,y)$

No, this is not true, everyone is not the same height.

(e) $\forall x\, \exists y\ P(x,y)$

Yes, true since (b) is true.

(f) $\forall y\, \exists x\ P(x,y)$

Yes, true since (c) is true.

4. State the converse, inverse, and contrapositive to the sentence: If yesterday was Sunday or tomorrow is Tuesday, then today is Monday. You are not allowed to accomplish this by placing: 'it is not true' in front of either the hypothesis, the conclusion, or the entire sentence.

Converse: If today is Monday, then yesterday was Sunday or tomorrow is Tuesday.

Inverse: If yesterday was not Sunday and tomorrow is not Tuesday, then today is not Monday.

Contrapositive: If today is not Monday, then yesterday was not Sunday and tomorrow is not Tuesday.

5. For which universe of discourses is the following quantified sentence true: $\forall y\, \exists x\ (y = x^2)$. Your choices are $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Q}^+$. You may assume both x and y belong to the same universe of discourse.

The sentence is not true for $\mathbb{R}$, since no x works for $y = -1 \in \mathbb{R}$.
The sentence is true for $\mathbb{R}^+$, since for any fixed $y > 0$, we can simply let $x = \sqrt{y} > 0$, and thus $x \in \mathbb{R}^+$.
The sentence is false for $\mathbb{N}$, since if we set $y = 3$, no value of $x \in \mathbb{N}$ works (in fact the only x we know work are $x = \pm\sqrt{3}$).
The sentence is false for $\mathbb{Q}^+$ by the same argument as for $\mathbb{N}$.

6. State, in argument form, the following valid arguments. (a) Modus Ponens, (b) Modus Tollens, and (c) Hypothetical Syllogism.

Modus Ponens	Modus Tollens	Hyp. Syl.
p	$\sim q$	$p \to q$
$p \to q$	$p \to q$	$q \to r$
$\therefore q$	$\therefore \sim p$	$\therefore p \to r$

7. Determine, by any method you so choose, if the following argument is valid.

$$\begin{array}{l} p \\ (p \vee q) \to r \\ q \to \sim r \\ \hline \therefore \sim q \end{array}$$

We know that Problem 1 gives the equivalency $((p \vee q) \to r) \Leftrightarrow ((p \to r) \wedge (q \to r))$. So the second assumption gives us $(p \to r) \wedge (q \to r)$. And this in turn gives us $(p \to r)$, and also $(q \to r)$. Using Modus Ponens on p and $(p \to r)$ gives r. Using contraposition on $q \to \sim r$ gives $r \to \sim q$. Applying Modus Ponens to r and $r \to \sim q$ gives $\sim q$. Thus, the argument is valid!

If one were so inclined (which we are not here), we could also prove that the following is a tautological sentence:

$$(p \wedge ((p \vee q) \to r) \wedge (q \to \sim r)) \to \sim q$$

8. Construct a sentence P of three variables p, q, and r whose truth table's final column is as given below:

p	q	r		P's final column
T	T	T		T
T	T	F		F
T	F	T		T
T	F	F		T
F	T	T		F
F	T	F		T
F	F	T		T
F	F	F		T

Note that there are only two rows which result in a final truth value of F, otherwise the result is T. If we consider the sentence $\sim p \vee \sim q \vee r$, this will result in a sentence whose only false final column answer will be in row 2. Similarly, the sentence $p \vee \sim q \vee \sim r$ will have only one false in its final truth table column, and it will be row 5. Thus, the conjunction of these two sentences will satisfy the requirement. I.e. setting P to $(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$ will work.