Math 3283 - Foundations of Mathematics

Exam 2 - 2019.04.05

Solutions

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1. What should you suppose in order to prove the proposition:

For any positive integers n and m, if at least one of n or m is even, then nm is not prime.

using each of the following methods of proof:

(a) direct proof.

For a direct proof, we assume the hypothesis: At least one of n or m is even.

(b) proof by contradiction.

For a proof by contradiction, we assume the hypothesis and the negation of the conclusion: At least one of n or m is even and nm is prime.

(c) proof by contraposition.

Lastly, we assume the negation of the conclusion: nm is prime.

2. Disprove the following conjecture:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ \left(0 < |x-1| < \delta \to \left|x^2+1\right| < \varepsilon\right)$$

First we negate the sentence:

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ \left(0 < |x-1| < \delta \wedge \left|x^2+1\right| \geq \varepsilon\right)$$

For disproof, we simply find an example which makes the negation true. Let us set $\varepsilon = \frac{1}{2}$. Then $\left|x^2+1\right| \geq 1 > \frac{1}{2}$ is always true for any $\delta > 0$. $\square$

3. Prove the following conjecture:

If n^2 is even, then n is even.

We prove this by contraposition, for which the statement to prove is:

If n is odd, then n^2 is odd.

By assumption, n is odd means $\exists k \in \mathbb{N}$ s.t. $n = 2k+1$. Thus, $n^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1$. Since $l = 2k^2+2k \in \mathbb{N}$, we have that $n^2 = 2l+1$ which is the definition of n^2 being odd. $\square$

4. Recall that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Furthermore, the sum of the first n odd integers can be expressed by the sum: $\displaystyle\sum_{k=1}^{n} 2k-1$. Prove by induction the following conjecture:

For any positive integer n, the sum of the first n odd integers is the perfect square n^2.

So by induction, we will first do the base case of $n = 1$.

$$\sum_{k=1}^{1} 2k - 1 = 1 = 1^2.$$

Next, we assume that $\sum_{k=1}^{n} 2k - 1 = n^2$ and try to show that $\sum_{k=1}^{n+1} 2k - 1 = (n+1)^2$

$$\begin{aligned}\sum_{k=1}^{n+1} 2k - 1 &= \left(\sum_{k=1}^{n} 2k - 1\right) + 2(n+1) - 1 \\ &= \left(\sum_{k=1}^{n} 2k - 1\right) + 2(n+1) - 1 \\ &= n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \qquad \square\end{aligned}$$

5. Prove the following conjecture by the method of cases:

$\forall x \in \mathbb{R}\ x + |x - 3| \geq 3$.

We proceed by cases. Case (1) will be $x \geq 3$, and case (2) will be $x < 3$.

Case (1): If $x \geq 3$, then $|x - 3| = x - 3$. Thus $x + |x - 3| = x + x - 3 = 2x - 3$. But since $x \geq 3$ for this case, $2x - 3 \geq 2 \cdot 3 - 3 = 3$.

Case (2): If $x < 3$, then $|x - 3| = 3 - x$. Thus $x + |x - 3| = x + 3 - x = 3$ and therefore $x + |x - 3| \geq 3$. $\square$