Math 3283 - Foundations of Mathematics
Final Exam - 2019.05.06

Name: ______________________________

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1. Construct a truth table for the sentence: $[p \wedge (q \to r)] \longleftrightarrow [(p \wedge q) \to r]$

2. Determine if each of the quantified sentences is true or false. If it is false, explain and give an example of why it is false. You may assume that the universe of discourse is $\mathbb{R}$.

(a) $\forall x \, \exists y \;\; |x+1| = y$

(b) $\forall y \, \exists x \;\; |x+1| = y$

3. Given $p \to q$ state the corresponding inverse, converse, and contrapositive sentences.

4. Express the negation of $p \to q$ without using parentheses.

5. Prove $5^n < n!$ for $n \geq 12$ by weak induction.

6. Prove every positive integer n can be written as the sum of *distinct* powers of 2 using strong induction. As examples, $7 = 2^2 + 2^1 + 2^0$, and $12 = 2^3 + 2^2$. *Hint: Arbitrarily subtract off the largest possible power of 2 first.*

7. Prove that $5n + 6$ is odd iff n is odd.

8. Prove for any positive integer n, $3n^2 - n + 4$ is even.

9. The universe of discourse for this problem will be $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and consider the following sets: $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{0, 2, 4, 6, 8, 10\}$, and $C = \{0, 3, 5, 7, 9, 10\}$. Compute each of the following:
(a) $A \cup B$
(b) $A \cap B$
(c) $\overline{C}$
(d) $(A \cap C) \cap \overline{B}$
(e) $A \cap B \subseteq C$
(f) $\overline{A} \subseteq B \cup C$
(g) $A - B$
(h) $B - C$
(i) $\mathcal{P}(A - C)$

10. Using the pick-a-point method, prove the following set theoretical statement: $A \subseteq B \wedge C \subseteq D \to A \cap C \subseteq B \cap D$.

11. Prove the following set theoretical theorem: $B - (B - A) = A \cap B$, using set identities from the following list:
Law 1: $A - B = A \cap \overline{B}$
Law 2: $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Law 3: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
Law 4: $\overline{\overline{A}} = A$
Law 5: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Law 6: $A \cap \overline{A} = \{\}$
Law 7: $A \cup \{\} = A$
Law 8: $A \cup B = B \cup A$

12. The definition of a relation R being *connected* is $\forall x, y \; x \neq y \rightarrow xRy \vee yRx$. We set $A = \{1, 2, 3, 4\}$, and define the relation on $A \times A$ as $R = \{(1, 2), (2, 1), (1, 3), (1, 4), (3, 2), (4, 2)\}$.
(a) Explain why R is not connected on the set A.
(b) What elements would we have to add to R to make it connected?

13. Explain why the relation of $<$ on the set of all real numbers ($\mathbb{R}$) is connected.

14. Consider the set of all human beings on planet earth, and the following relation L. We define xLy to be true between persons x and y iff person x lives within a mile of person y. Determine if the relation L is any of (a) reflexive, (b) symmetric, or (c) transitive, on the set of all human beings on earth.