Logic

Alfred Tarski
updated by Karl Frinkle

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Preface for New Edition

I should start off by saying that this text is, in this mathematician's opinion, the best, basic introduction to logic textbook available. Many current logical textbooks fail to focus on the most important aspect of logic to the sciences, which is, the deductive method. The rigorous approach to building a deductive theory based on an axiomatic system, and the consequences of selecting axioms and primitive terms is thoroughly, and carefully explained, yet readily accessible to the reader.

Many texts spend too much time dealing with logical symbolism, truth tables, and clever word puzzles, while others tend to be only accessible to those studying the fundamental semantics and intricacies of logic. This text is the perfect balance between sufficiently deep logical theory and the application of this theory to algebraic axiomatic systems.

One might thus ask: Why would you wish to edit or alter such a text if you praise it so highly? The text itself has been 'updated' several times, the last being in 1994. In the last two updates, very little has been done to make the language of the writing or the symbolism more current. Furthermore, after listening to the comments of the students in my Honor's Logic Courses, I realized that the students tolerated the message of the book, just not the style in which that message was presented. Most textbooks today focus heavily on highlighting key items such as definitions, tables, laws, theorems, examples, etc... I have attempted to make the text more easily parsed upon reading.

One of the more important aspects of the upgrade to this text is in the rigorous, formal proof system-style approach to proving many of the theorems presented in this text. All previous versions of this text were very quick to wave hands at many of the more subtle steps required to rigorously prove many of the theorems given. In doing so, many logical laws and clever insights were missed, and I have attempted to include them in the details added.

I have also taken the liberty of either incorporating optional sections of the text (cordoned off by *'s) or removing them. After using this text numerous times, it became clear which material really had an impact on the overall theme of the course, and what material was either not necessary or added confusion to the learning process. In a similar fashion, some of the homework problems have been reworked. Less emphasis has been put on geometrically oriented problems, and most of the questions starting off with "Give examples from..... of such and such a property..." have simply been removed or replaced.

I have found Dr. Tarski's writing style to be rather refreshing and conversational in style, and have therefore attempted to leave most of his words intact. Any additions or further explanations hopefully blend in as seamlessly as I intended.

At this point, I need to give a special thanks to two very dedicated students: Laiken McMurl and Michaela Metts, who were students in my Fall 2015 Logic course. These two students are the definition of ideal students, they read the text multiple times, had intelligent discourse over all of the material, and critiqued every aspect of the book. Many of the modifications done in the text were the direct result of input from these two students. For instance, the compilation of rules, laws, and logical identities found after the set of prefaces to this text was one important suggestion they made which I feel will benefit all serious students of this text. Included in this list are some clever logical identities that may help solve some of the later exercises and are not discussed in the text.

Added in 2019: After a few years of teaching with the modified text, I decided to add a chapter focused on the topics of consistency and completeness, which were barely touched upon in the original text. Chapter 7 is based on parts of Geoffrey Hunter's *Metalogic: An Introduction to the Metatheotry of Standard First Order Logic*, which is, in the opinion of the writer, an excellent followup to this text. Additionally, the book has now been broken up into three parts. Chapters 6 and 7 are now 'Deductive Method and Formal Systems'. Last, and certainly not least, I would like to thank Fisher Fulton for being the test subject for a second semester of logic, which led to the addition of the aforementioned new chapter . Her input, insights, and critiques were an important part of making this new chapter possible.

Karl Frinkle

Southeastern Oklahoma State University
Durant, OK, Some Month, Some year

Preface

The present book is a partially modified and extended edition of my book *On Mathematical Logic and the Deductive Method*, which appeared first in 1936 in Polish and then in 1937 in an exact German translation (under the title: *Einführung in die mathematische Logik and in die Methodologie der Mathematik*). In its original form it was intended as a popular scientific book; its aim was to present to the educated layman – in a manner which would combine scientific exactitude with the greatest possible intelligibility – a clear idea of that powerful trend of contemporary thought which is concentrated about modern logic. This trend arose originally from the somewhat limited task of stabilizing the foundations of mathematics. In its present phase, however, it has much wider aims. For it seeks to create a unified conceptual apparatus which would supply a common basis for the whole of human knowledge. Furthermore, it tends to perfect and sharpen the deductive method, which in some sciences is regarded as the sole permitted means of establishing truths, and indeed in every domain of intellectual activity is at least an indispensable auxiliary tool for deriving from accepting assumptions.

The response accorded to the Polish and German editions, and especially some suggestions made by reviewers, gave rise to the idea of making the new edition not merely a popular scientific book, but also a textbook upon which an elementary college course in logic and the methodology of deductive sciences could be based. The experiment seemed the more desirable in view of a certain lack of suitable elementary textbooks in this domain.

Some very fundamental questions and notions were entirely passed over or merely touched upon in the previous editions, either because of their more technical character, or in order to avoid points of a controversial nature. As examples may be cited such topics as the difference between the usage of certain logical motions in systematic developments of logic and in the language of everyday life, the general method of verifying the laws of the sentential calculus, the necessity of a sharp distinction between words and their names, the concepts of the universal class and the null class, the fundamental notions of the calculus of relations, and finally the conception of the methodology as a general science of sciences. In the present edition all these topics are discussed (although not all in an equally thorough manner), since it seemed to me that to avoid them would constitute an essential gap in any textbook of modern logic. Consequently, the chapters of the first, general part of the book have been more or less extended; in particular, Chapter 2, which is devoted to the sentential calculus, contains much new material. I have also added many new exercises to these chapters, and have increased the number of historical indications.

While in the previous editions the use of special symbols was reduced to a minimum, I considered it necessary in the present edition to familiarize the reader with the elements of logical symbolism. Nevertheless, the use of this symbolism in practice remains very restricted, and is limited mostly to exercises.

In previous editions the principal domain from which examples were drawn for illustrating general and abstract considerations was high-school mathematics; for it was, and still is, my opinion that elementary mathematics, and especially algebra, because of the simplicity of its concepts and the uniformity of its methods of inference, is peculiarly appropriate for exemplifying various fundamental phenomena of a logical and methodological nature. Nevertheless, in the present edition, particularly in the newly added passages, I draw examples more frequently from other domains, especially from everyday life.

Independent of these additions, I have rewritten certain sections whose mastery by students has been found somewhat difficult.

The essential features of the book remain unchanged. The preface to the original edition, the major part of which is reprinted in the next few pages, will give the reader an idea of the general character of the book. Perhaps, however, it is desirable to point out explicitly at this place what the reader should not expect to find in it.

First, the book contains no systematic and strictly deductive presentation of logic; such a presentation would obviously not lie within the framework of an elementary textbook. It was originally my intention to include, in the present edition, an additional character entitled *Logic as a Deductive Science*, which – as an illustration of the general methodological remarks contained in Chapter 6 – would outline a systematic development of some elementary parts of logic. For a number of reasons this intention could not be realized; but I hope that several new exercises on this subject included in Chapter 6 will to some extent compensate for this omission.

Secondly, apart from two rather short passages, the book gives no information about the traditional Aristotelian logic, and contains no material drawn from it. But I believe that the space here devoted to traditional logic has been reduced in modern science; and I also believe that this opinion will be shared by most contemporary logicians.

And, finally, the book is not concerned with any problems belonging to the so-called logic and methodology of empirical sciences. I must say that I am inclined to doubt whether any special "logic of empirical sciences", as opposed to logic in general or the "logic of deductive sciences", exists at all (at least so long as the word "logic" is used as is present in this book – that is to say, as the name of a discipline which analyzes the meaning of the concepts common to all the sciences, and establishes the general laws governing the concepts). But this is rather a terminological, than a factual, problem. At any rate the methodology of empirical sciences constitutes an important domain of scientific research. The knowledge of logic is of course valuable in the study of this methodology, as it is in the case of any other discipline. It must be admitted, however, that logical concepts and methods have not, up to present, found any specific or fertile applications in this domain. And it is at least possible that this situation is not merely a consequence of the present stage of methodological researches. It arises, perhaps, from the circumstance that, for the purpose of an adequate methodological treatment, an empirical science may have to be considered, not merely as a scientific theory – that is, as a system of asserted statements arranged according to certain rules, – but rather as a complex consisting partly of such statements and partly of human activities. It should be added that, in striking opposition to the high development of the empirical sciences themselves, the methodology of these sciences can hardly boast of comparably definite achievements – despite the great efforts that have been made. Even the preliminary task of clarifying the concepts involved in this domain has not yet been carried out in a satisfactory way. Consequently, a course in the methodology of empirical sciences must have a quite different character from one in logic and must be largely confined to evaluations and criticisms of tentative gropings and unsuccessful efforts. For these and other reasons, I see little rational justification for combining the discussion of logic and the methodology of empirical sciences in the same college course.

A few remarks concerning the arrangement of the book and its use as a college text.

The book is divided into two parts. The first gives a general introduction to logic and the methodology of deductive sciences; the second shows, by means of a concrete example, the sort of applications which logic and methodology find in the construction of mathematical theories, and thus affords an opportunity to assimilate and deepen the knowledge acquired in the first part. Each chapter is followed by appropriate exercises. Brief historical indications are contained in footnotes.

Passages, and even whole sections, which are set off by asterisks "$\star$" both at the beginning and at the end, contain more difficult material, or presuppose familiarity with other passages containing such material; they can be omitted without jeopardizing the intelligibility of subsequent parts of the book. This also applies to the exercises whose numbers are preceded by asterisks.

I feel that the book contains sufficient material for a full-year course. Its arrangement, however, makes it feasible to use it in half-year courses as well. It used as a text in half-year logic courses in a department of philosophy, I suggest a thorough study of its first part, including the more difficult portions, with the entire omission of the second part. If the book is used in a half-year course in a mathematics department – for instance, in the foundations of mathematics

–, I suggest the study of both parts of the book, with the omission of the more difficult passages.

In any case, I should like to emphasize the importance of working out the exercises carefully and thoroughly; for they not only facilitate the assimilation of the concepts and principles discussed, but also touch upon many problems for the discussion of which the text provided no opportunity.

I shall be very happy if this book contributes to the wider diffusion of logical knowledge. The course of historical events has assembled in this country the most eminent representatives of contemporary logic, and has thus created especially favourable conditions for the development of logical thought. These favourable conditions can, of course, be easily overbalanced by other and more powerful factors. It is obvious that the future of logic, as well as of all theoretical science, depends essentially upon normalizing the political and social relations of mankind, and thus upon a factor which is beyond the control of professional scholars. I have no illusions that the development of logical thought, in particular, will have a very essential effect upon the process of the normalization of human relationships; but I do believe that the wider diffusion of the knowledge of logic may contribute positively to the acceleration of this process. For, on the one hand, by making the meaning of concepts precise and uniform in its own field and by stressing the necessity of such a precision and uniformization in any other domain, logic leads to the possibility of better understanding between those who have the will to do so. And, on the other hand, by perfecting and sharpening the tools of thought, it makes people more critical – and thus makes less likely their being misled by all the pseudo-reasonings to which they are in various parts of the world incessantly exposed to today.

I gratefully acknowledge my indebtedness to Dr. O. Helmer, who performed the translation of the German edition into English. I want to express my warmest gratitude to Dr. A. Hofstadter, Mr. L. K. Krader, Professor E. Nagel, Professor W. V. Quine, Mr. M. G. White, and especially Dr. J. C. C. McKinney, and Dr. P. P. Wiener, who were unsparing in their advice and assistance while I was preparing the English edition. I also owe many thanks to Mr. K. J. Arrow for his help in reading proofs.

Alfred Tarski

Harvard University September 1940

The present book is a photographic reprint of the first English edition, and no large-scale changes could be introduced in it. Misprints, however, have been corrected, and a number of improvements in detail have been made. I wish to thank readers and reviewers for the helpful suggestions, and I am especially indebted to Miss Louise H. Chin for her assistance in preparing the present edition for publication.

A. T.

University of California
Berkeley, August 1945

Original Edition Preface

In the opinion of many laypersons, mathematics is today already a dead science: after having reached an unusually high degree of development, it has become petrified in rigid perfection. This is an entirely erroneous view of the situation; there are but few domains of scientific research which are passing through a phase of such intensive development at present as mathematics. Moreover, this development is extraordinarily manifold: mathematics is expanding its domain in all possible directions, it is growing in height, in width, and in depth. It is growing in height, since, on the solid of its old theories which look back upon hundreds if not thousands of years of development, new problems appear again and again, and ever more perfect results are being achieved. It is growing in width, since its methods permeate other branches of sciences, while its domain of investigation embraces increasingly more comprehensive ranges of phenomena and ever new theories are being included in the large circle of mathematical disciplines. And finally it is growing in depth, since its foundations become more and more firmly established, its methods perfected, and its principles stabilized.

It has been my intention in this book to give those readers who are interested in contemporary mathematics, without being actively concerned with it, at least a very general idea of that third line of mathematical development, i.e. its growth in depth. My aim has been to acquaint the reader with the more important concepts of a discipline which is known as mathematical logic, and which has been created for the purpose of a firmer and more profound establishment of the foundations of mathematics; this discipline, in spite of its brief existence of barely a century, has already attained a high degree of perfection and plays today's role in the totality of our knowledge that far transcends its originally intended boundaries. It has been my intention to show that the concepts of logic permeate the whole of mathematics, that they comprehend all specifically mathematical concepts as special cases, and that logical laws are constantly applied – be it consciously or unconsciously – in mathematical reasonings. Finally, I have tried to present the most important principles in the construction of mathematical theories – principles which form the subject matter of still another discipline, the methodology of mathematics – and to show how one sets about using the principles in practice.

It has not been easy to carry this whole plan through within the framework of a relatively small book without presupposing on the part of the reader and specialized mathematical knowledge or any specific training in reasonings of an abstract character. Throughout the book a combination of the greatest possible intelligibility with the necessary conciseness had to be attempted, with a constant care for avoiding errors or cruder inexactitudes from the scientific standpoint. A language had to be used which deviates as little as possible from the language of everyday life. The employment of a special logical symbolism had to be given up, although this symbolism is an invaluable tool which permits us to combine conciseness with precision, removes to a large degree the possibility of ambiguities and misunderstandings, and is thereby of essential service in all subtler considerations. The idea of a systemic treatment had to be abandoned from the beginning. Of the abundance of the questions which present themselves only a few could be discussed in detail, others could only be touched on superficially, while still others had to be passed over entirely, with the consciousness that the selection of topics discussed would inevitably exhibit a more or less arbitrary character. In those cases in which contemporary science has as yet not taken any definite stand and offers a number of possible and equally correct solutions, it was out of the question to present objectively all known views. A decision in favour of a definite point of view had to be made. When making such a decision I have taken care, not primarily to have it conform to my personal inclinations, but rather to choose a method of solution which would be as simple as possible and which would lend itself to a popular mode of presentation.

I do not have the illusion that I have entirely succeeded in overcoming these and other difficulties.

Compilation of Rules and Laws

To help the reader keep track of all the possible rules of inference, logical laws, and logical equivalencies that may come in handy throughout this text, we have compiled a running list.

Logical Rules of Inference

Rule of Substitution. *If a sentence of a universal character, that has already been accepted as true, contains sentential variables, and if these variables are replaced by other sentential variables or by sentential functions or sentences – always substituting equal expressions for equal variables throughout –, then the sentence obtained in this way may be recognized as true.*

Rule of Detachment (Modus Ponens). *If two sentences are accepted as true, of which one has the form of an implication while the other is the antecedent of this implication, then the sentence which forms the consequent of the implication may also be recognized as true.* (*We* detach *thus, so to speak, the antecedent from the whole implication.*).

Rule of And, Joining Together. *If any two sentences are accepted as true, then their conjunction may be recognized as true.*

Rule of Replacement. *If, in a certain context, a formula having the form of an equation, e.g.:*

$$x = y,$$

has been assumed or proved, then it is permissible to replace, in any formula or sentence occurring in this context, the left side of the equation by its right side, e.g. "x" by "y", and conversely.

Logical Laws

Law of And, Breaking Apart. $(p \wedge q) \to p$

Law of And, Breaking Apart′. $(p \wedge q) \to q$

Law of Identity. $p \to p$

Law of Or, Joining Together. $p \to q \vee p$

Law of Hypothetical Syllogism. $[(p \to q) \wedge (q \to r)] \to (p \to r)$

Law of Biconditional. $[(p \to q) \wedge (q \to p)] \to (p \leftrightarrow q)$

Law of Contradiction. $\sim [p \wedge (\sim p)]$.

Law of Excluded Middle. $p \vee (\sim p)$.

Law of And Tautology. $(p \wedge p) \leftrightarrow p$.

Law of Or Tautology. $(p \vee p) \leftrightarrow p$.

Law of Commutative And. $(p \wedge q) \leftrightarrow (q \wedge p)$.

Law of Commutative Or. $(p \vee q) \leftrightarrow (q \vee p)$.

Law of Associative And. $[(p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$.

Law of Associative Or. $[(p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$.

Law of Contraposition. $(p \to q) \to (\sim q \to \sim p)$

Law of Double Negation. $\sim (\sim p) \to p$

Law of Reductio ad Absurdum. $[p \to (\sim p)\,] \to (\sim p)$

Law of General Contradiction (Modus Tollens). $[\,(p \to q) \wedge \sim q\,] \to \sim p$

Law of Disjunctive Syllogism. $[\,(p \vee q) \wedge \sim p\,] \to \sim q$

Law of Constructive Dilemma. $[\,(p \to q) \wedge (r \to s) \wedge (p \vee r)\,] \to (q \vee s)$

Law of Destructive Dilemma. $[\,(p \to q) \wedge (r \to s) \wedge (\sim p \vee \sim s)\,] \to (\sim p \vee \sim r)$

Law of Bidirectional Dilemma. $[\,(p \to q) \wedge (r \to s) \wedge (p \vee \sim s)\,] \to (q \vee \sim r)$

Law of Composition. $[\,(p \to q) \wedge (p \to r)\,] \to (p \to (q \wedge r))$

Law of Exportation. $[\,(p \wedge q) \to r\,] \to [\,p \to (q \to r)\,]$

Law of Importation. $[\,p \to (q \to r)\,] \to [\,(p \wedge q) \to r\,]$

Useful Tautological Statements

Besides the rules and laws listed previously, there are some other important tautological statements that are used throughout this text, most of these appear in the homework problems.

(1) $[\sim(\sim p)] \leftrightarrow p$

(2) $[\sim(p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$

(3) $[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$

(4) $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

(5) $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

(6) $p \rightarrow (q \rightarrow p)$

(7) $(\sim p) \rightarrow (p \rightarrow q)$

(8) $(p \rightarrow q) \vee (q \rightarrow p)$

(9) $[(p \rightarrow q) \wedge (p \rightarrow r)] \leftrightarrow [p \rightarrow (q \vee r)]$

(10) $[(p \wedge q) \rightarrow r] \rightarrow [(p \rightarrow r) \vee (q \rightarrow r)]$

(11) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

(12) $(p \vee q \vee r) \leftrightarrow [\sim p \rightarrow (q \vee r)]$

(13) $(p \vee q \vee r) \leftrightarrow [(\sim p \wedge \sim q) \rightarrow r]$

(14) $(p \wedge \sim p) \leftrightarrow \mathrm{F}$

(15) $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

(16) $(p \vee F) \leftrightarrow p$

(17) $(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$

(18) $(p \rightarrow q) \leftrightarrow [(p \vee q) \rightarrow q]$

(19) $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

(20) $(p \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow \sim q)$

(21) $(p \wedge q) \rightarrow (p \vee q)$ (Deveb's Law)

(22) $[(p \rightarrow r) \wedge (q \rightarrow r)] \leftrightarrow [(p \vee q) \rightarrow r]$

(23) $\text{True}_1 \rightarrow \text{True}_2$

(24) $\forall x, y\ P(x, y) \rightarrow \forall x\ P(x)$

(25) $(p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]$

(26) $(p \leftrightarrow q) \rightarrow \{[(r \wedge p) \rightarrow s] \rightarrow [(r \wedge q) \rightarrow s]\}$

(27) $(p \leftrightarrow q) \to \{(q \to r) \to (p \to r)\}$

(28) $(p \to q) \to \{(p \to r) \to [p \to (q \wedge r)]\}$

(29) $(p \to \sim q) \to (q \to \sim p)$

(30) $(p \to q) \to [(q \to r) \to (p \to r)]$

Part I

Elements of Logic

Chapter 1

On The Use of Variables

1.1 Constants and Variables

Every scientific theory is a system of sentences which are accepted as true and which may be called *Laws* or *Asserted Statements*. In mathematics, these sentences follow one another in a definite order according to certain principles which will be discussed in Chapter 6. The method in which each new statement follows from the previous statements arises from the application of logic, which will be studied extensively in this text. This application of the logical processes to statements in the realm of mathematics constitutes what is known as a *proof*, which in themselves establish the validity of *Theorems*. We formally introduce the following definition:

Definition 1.1. A *sentence* is a statement which can be determined to be either true or false.

A *sentence* need not always be true, or always be false. For instance, the sentence "Today is Wednesday", can be determined to be true or false whenever it is read or spoken. However, the only way that the sentence is true is if it analyzed on a Wednesday.

We will encounter many terms and symbols while we explore mathematics theorems and proofs in this text, and most fall into two categories, those being *constants* and *variables*

Definition 1.2. *Constants* are those terms and symbols which have well-determined meanings which remain unchanged as the field of mathematics is developed.

Examples of constants would be "*numbers*", or specific numbers such as "*zero*" ("0") and "*one*" ("1"), as well as symbols "+" and "·", which represent the arithmetic operations of "*addition*" and "*multiplication*", respectively.[1]

Logic, New New Edition.
Karl Frinkle.

Definition 1.3. A *variable* is a term or symbol, usually denoted by single letters such as "a", "x", or "α", which do not possess any meaning themselves, but have meaning when constants replace them in statements.

For example, the sentence:

x is an integer.

is not a sentence, as it cannot be determined to be true or false. If, however, we replace the *variable* x with the *constant* 3, we arrive at the true sentence "3 is an integer". However, if x is replaced with the value of $\frac{1}{2}$, the sentence "$\frac{1}{2}$ is an integer" is false.

1.2 Expressions Containing Variables

As stated in the first section, variables do not have a meaning by themselves, so the statement "x an integer" has no meaning, although the statement does indeed have the grammatical form of a sentence. When x is replaced by "3" or "$\frac{1}{2}$", the statement expresses a definite assertion, thus becoming a sentence.

Definition 1.4. An expression which contains variables and, on replacement of these variables by constants, becomes a sentence, is called a *sentential function.*

It should be noted that there is a difference between the mathematical definition of a function, which we will talk about in Chapter 5, and that of a sentential function. For instance, the equation

$$x^2 + y^2 = 1 \tag{1.1}$$

is a sentential function, since it becomes a sentence upon replacement of variables x and y by real numbers. However, equation (1.1) does not satisfy the mathematical definition of a function, and is usually referred to as a *formula* when an equal sign is present. It should be noted that we will refer to sentential functions as sentences in certain circumstances, but only in cases where there is no danger of any misunderstanding.

One can also construct sentential functions which are in no way related to mathematical construct. For instance, consider

$$x \textit{ is the brother of } y. \tag{1.2}$$

Clearly substituting number for the variables x and y in the above function results in a nonsensical statement. However, if we replace x by a name such as "Jim" and y by "Robert", the resulting sentence can be determined to be true or false.

The role of variables in a sentential function can be compared to that of blanks left in a questionnaire; just as the questionnaire acquires a definite content only after the blanks have been filled in, a sentential function becomes a sentence only after constants have been inserted in place of the variables.

Definition 1.5. If the result of the replacement of the variables in a sentential function by constants leads to a true sentence, then we say that the given constants *satisfy* the sentential function.

As examples, the numbers 1, 2, and $\frac{1}{2}$ satisfy the sentential function

$$x < 3,$$

but the numbers 3, 4 and 4.5 do not.

Besides the sentential functions, there is another class of expressions containing variables meriting our attention.

Definition 1.6. *Designatory functions* (or *descriptive functions*) are expressions, which, on replacement of variables by constants, turn into designations (or descriptions) of objects.

As an example, the expression

$$2x + 1$$

is a designatory function, because we obtain the designation of a specific number, if in it we replace the variable x by an arbitrary numerical constant (i.e. a constant denoting a number).

Among the designatory functions occurring in arithmetic, we have, in particular, all the so-called algebraic expressions which are composed of variables, numerical constants and symbols of the four fundamental arithmetic operations, such as:

$$x - y, \quad \frac{x+1}{y+2}, \quad 2 \cdot (x + y - z).$$

Algebraic equations can now be described by two expressions connected by the symbol "=", and are therefore sentential functions. Sometimes the expressions on either side of the "=" symbol are designatory functions, but they can also be constants. As far as equations are concerned, a special terminology has become customary in mathematics; thus the variables occurring in the equation are referred to as the *unknowns*, and the numbers satisfying the equation are called *roots* or *solutions* of the equation. E.g., in the equation

$$x^2 + 6 = 5x$$

the variable "x" is the unknown, while the numbers 2 and 3 are roots(solutions) of the equation.

Of the variables "x", "y", ... employed in arithmetic, it is said that they *stand for designations of numbers* or that numbers are *values* of these variables. Already alluded to, what this really means is that a sentential function containing the symbols "x", "y", ... becomes a sentence, if these symbols are replaced by such constants as designate numbers, and not by expressions designating operations on numbers, relations between numbers, or even objects outside the field of arithmetic like geometrical configurations, animals, plants, etc. Likewise, the variables occurring in geometry stand for designations of points and geometrical figures.

1.3 Formation of Sentences by Means of Variables

Apart from the replacement of variables by constants, there is still another way in which sentences can be obtained from sentential functions. Let us consider the formula:

$$x + y = y + x$$

It is a sentential function containing the two variables "x" and "y" that is satisfied by an arbitrary pair of numbers; if we put any numerical constants in place of "x" and "y", we always obtain a true formula. We express this fact in the following sentence:

For any numbers x and y, $x + y = y + x$.

The expression given above is already a sentence, and true as well. (This happens to be the commutative property of arithmetic). Most of the important laws of mathematics are defined similarly, namely, all so-called *universal sentences*, which assert that arbitrary objects of a certain category (or set), such as real numbers, have specific properties. Often, in the formulation of universal sentences, the phrase "*for any x and y...*", is often omitted, as it is assumed. Thus, the above sentence is also equivalent to the sentence

$$x + y = y + x.$$

This has been a well accepted usage, to which we shall generally adhere in this text.

Let us now consider the sentential function

$$x > y + 1. \tag{1.3}$$

The formula (1.3) fails to be satisfied by all pairs (x, y) of real numbers. For instance, replacing the variables x and y by 3 and 4, respectively, yields the false sentence $3 > 5$. Thus, assuming the well accepted usage previously discussed, formula (1.3) is false. However, it should also be noted that there are constants for which replacing the variables x and y with said constants makes the formula true. For instance, if x is 3 and y is 1, we arrive at the relation $3 > 2$, which is true. This situation is expressed by the following phrase:

For some numbers x and y, $x > y + 1$, or equivalently,

There are numbers x and y such that $x > y + 1$.

The two previous sentences are examples of *existential sentences*, which state the existence of something satisfying specified conditions. We can now construct sentences from any given sentential function, without replacing variables with constants, by the use of phrases such as '*for any x*' or '*there is some y*', but it depends both on the phrases and the content of the sentential functions as to whether or not the resulting sentence is true.

As an example, consider the formula:

$$x = x + 1 \tag{1.4}$$

Clearly, adding the phrase '*for any x*' or '*there is some x*' does not result in a true sentence. We can further conclude through the rules of algebra that this formula does not hold for any constant value of x.

A sentence which does not have any universal or existential character must contain no variables is called a *singular sentence*, as illustrated in the following example:

$$3 + 2 = 4 + 1$$

Aside from absolutely universal and existential sentences, there exists sentences of a *conditionally existential* nature which involve both existential and universal quantified variables. Take for instance, the sentence

$$\textit{For any } x \textit{ and } y, \textit{ there is a number } z \textit{ such that } x = y + z, \tag{1.5}$$

which states the existence of a number z based on the condition that specific numbers x and y exist.

1.4 Universal and Existential Quantifiers, Free and Bound Variables

Phrases of the form

for any x, y, ...

and

there are x, y, ... such that ...

are called *quantifiers*; the former is said to be a *universal*, the latter an *existential* quantifier. Quantifiers are known as *operators*; however there are expressions which are considered to be operators which are not quantifiers. In Section 1.3 we attempted to explain the meaning of the two quantifiers. In order to emphasize their significance it may be pointed out that an expression containing variables is a sentence only through the explicit or implicit use of operators. Without the help of operators, the use of variables in the formulation of mathematical theorems would be excluded.

In everyday language, it is not customary to use variables, and quantifiers are also, for this reason, not in use. There are, however, certain words in general usage which exhibit a very close connection with quantifiers, namely words such as "*every*", "*all*", "*a certain*", and "*some*". The connection becomes obvious when we observe that expressions like:

all people are mortal

or

some people are wise

have about the same meaning as the following sentences, formulated with the help of quantifiers:

for any x, if x is a person, then x is mortal

or

there is an x, such that x is both a person and wise,

respectively.

For the sake of brevity, the quantifiers are replaced by symbolic expressions in mathematics.

Definition 1.7. The symbol $\forall$ is called the *universal quantifier*, and the symbolic expression $\forall\, x, y, \ldots$ is equivalent to the expression "for any x, y, … ", or "for all x, y, … ".

Definition 1.8. The symbol $\exists$ is called the *existential quantifier*, and the symbolic expression $\exists\, x, y, \ldots$ is equivalent to the expression "there exists x, y, … such that", or "there are x, y, … such that".

To illustrate the use of the symbolic quantifiers, the quantified sentence (1.5) of section 1.3 can now be expressed as

$$\forall\, x, y\ \exists\, z\ \ (x = y + z) \tag{1.6}$$

A sentential function in which the variables "x", "y", "z", … occur automatically becomes a sentence as soon as one prefixes to it one or more operators containing all those variables. If, however, some of the variables do not occur in the operators, the expression in question remains a sentential function, without becoming a sentence. For example, the formula

$$x = y + z \tag{1.7}$$

changes into a sentence if preceded by one of the following quantifiers:

$$\forall\, x, y, z$$
$$\exists\, x, y, z$$
$$\forall\, x, y\ \exists\, z$$

and so on. (How many combinations are there?). If, on the other hand, we prefix with the quantifiers

$$\forall\, x, y$$
$$\exists\, z$$
$$\forall\, y\ \exists\, z$$

we do not yet arrive at a sentence. The expression obtained by prefixing the middle quantifier from above with sentential function (1.7) yields the sentential function

$$\exists\, z\ (x = y + z), \tag{1.8}$$

which becomes a sentence only upon substituting constants for the two variables "x" and "y". Thus, we can see that among the variables that may occur in a sentential function, two different kinds can be distinguished.

Definition 1.9. Variables which restrict a sentential function from being a sentence are called *free variables.*

The occurrence of free variables is the decisive factor in determining that the expression under consideration is a sentential function and not a sentence. In order to effect the change from a sentential function to a sentence it is necessary to replace these variables with constants or else to put operators in front of the sentential function that contains those free variables. The remaining variables are given the following definition:

Definition 1.10. Variables which are prefixed by quantifiers, or are in some way regarded as constants, are said to be *bound variables.*

Bound variables are not altered by the transformation of the addition of quantifiers. In expression (1.6), the variables "x" and "y" are bound by the universal quantifier, and "z" by an existential quantifier. Since all variables are bound, the expression is a sentence. In expression (1.7), all variables are free, and thus we are left with a sentential function. Lastly, expression (1.8) has only one bound variable, "z", which is bound by an existential quantifier. The variables "x" and "y" are free.

It depends entirely upon the structure of the sentential function, namely, upon the presence and position of the operators, whether any particular variable occurring in it is free or bound. This may be best seen by means of a concrete example. Let us, for instance, consider the following sentential function:

$$\forall\, x\ (\text{if } x = 0 \text{ or } y \neq 0,\ \text{then } \exists\, z\ (x = y \cdot z)), \tag{1.9}$$

This function begins with a universal quantifier containing the variable "x", and therefore, the latter, which occurs three times in this function, occurs at all these places as a bound variable. The first instance of "x" makes up part of the quantifier, while at the other two places it is, as we say, *bound by the quantifier.* The situation is similar for the variable "z", for even though the initial quantifiers in the expression do not contain the variable "z", the second half of the expression starts off with an existential quantifier binding "z". Since there is no quantifier which binds the variable "y", it is free in both places in which it appears.

The fact that quantifiers bind variables – that is, that they change free into bound variables in the sentential function which follow them – constitutes a very essential property of quantifiers. Several other expressions are known which have an analogous property; with some of them we shall become acquainted later (see Sections 3.5 and 4.2), while some others play important roles in higher mathematics. The term *operator* is the general term used to denote all expressions having this property.

1.5 The Importance of Variables in Mathematics

As we have seen in Section 1.3, variables play a leading role in the formulation of mathematical theorems. From what has been said, it does not follow, however, that it would be impossible in principle to formulate the latter without the use of variables since even comparatively simple sentences would assume a complicated and obscure form. As an illustration, let us consider the following theorem of arithmetic:

Theorem.

$$\forall x, y \ \left(x^3 - y^3 = (x - y) \cdot (x^2 + xy + y^2) \right).$$

Without the use of variables, this theorem would look as follows:

Theorem. *The difference of the third powers of any two numbers is equal to the product of the difference of these numbers and a sum of three terms, the first of which is the square of the first number, the second the product of the two numbers, and the third the square of the second number.*

An even more essential significance, from the standpoint of the economy of thought, attaches to variables as far as mathematical proofs are concerned. This fact will be readily confirmed if one attempts to eliminate the variables in any of the proofs which will be encountered later. In particular, the solution to many mathematical problems was only possible after the introduction of variables and quantifiers. Without exaggeration it can be said that the invention of variables constitutes a turning point in the history of mathematics; with these symbols, we acquired a tool that prepared the way for the tremendous development of the mathematical science and for the solidification of its logical foundations.[2]

Exercises

1. Determine which of the following expressions are sentential functions, and which are designatory functions:
   (a) x is the capital of Oklahoma
   (b) $7 - x = 2$
   (c) The capital of the state z
   (d) The number y is greater than 4
   (e) The sum of two numbers u and v
   (f) The sum of two numbers u and v is positive.

2. The sentential functions which are encountered in arithmetic and which contain only one variable (which may occur in several different places in the given sentential function) can be divided into three categories: functions satisfied by every number; functions satisfied by some numbers, but not all; functions not satisfied by any number. Determine to which of the three categories each of the following sentential functions belong:
   (a) $x + 2 = 5 + x$
   (b) $x^2 = 49$
   (c) $(y + 2) \cdot (y - 2) < y^2$
   (d) $y + 24 > 36$
   (e) $z = 0$ or $z < 0$ or $z > 0$
   (f) $z + 24 > z + 36$

3. By writing quantifiers containing the variables "x" and "y" in front of the sentential function: $x > y$, it is possible to obtain various sentences from it, for instance

$$\forall\, x, y\ (\, x > y\,)$$
$$\forall\, x\ \exists\, y\ (\, x > y\,)$$
$$\exists\, y\ \forall\, x\ (\, x > y\,)$$

Formulate the remaining three quantified sentences, and determine which of them are true.

4. State a sentence of everyday language which has the same meaning as

For every x, if x is a dog, then x has a good sense of smell and that contains no quantifiers or variables.

5. Differentiate in the following expressions, between the free and bound variables:
   (a) x is divisible by y
   (b) for any x, $x - y = x + (-y)$
   (c) if $x < y$, then there is a number z such that $x < y$ and $y < z$
   (d) for any number y, if $y > 0$, then there is a number z such that $x = y \cdot z$
   (e) if $x = y^2$ and $y > 0$, then for any z, $x > -z^2$
   (f) if there exists a y such that $x > y^2$, then for any z, $x > -z^2$

6. Formulate the above expressions by replacing the quantifiers by the symbols introduced in Section 1.4.

7. If, in the sentential function (e) of Exercise 5 we replace the variable "z" in both places by "y", we obtain an expression in which "y" occurs in some places as free and in others as a bound variable; in what places and why?

8. Using your answer to Exercise 7, state quite generally under which conditions a variable occurs at a certain place of a given sentential function as a free or as a bound variable.

9. Determine which of the following quantified statements are true:
   (a) $\forall x\ \exists y,\ x \cdot y > 1$
   (b) $\exists x\ \forall y,\ x \cdot y > 1$
   (c) $\forall x, y,\ x \cdot y > 1$
   (d) $\exists x, y,\ x \cdot y > 1$
   (e) $\forall x\ \exists y,\ x^2 + y^2 > 1$
   (f) $\exists x\ \forall y,\ x^2 + y^2 > 1$
   (g) $\forall x\ \exists y,\ x^2 + y^2 < 1$
   (h) $\exists x\ \forall y,\ x + y^2 < 1$
   (i) $\forall x\ \exists y,\ x + y^2 > 1$
   (j) $\exists x\ \forall y,\ x + y^2 > 1$
   (k) $\forall x\ \exists y,\ x + y^2 < 1$
   (l) $\exists x\ \forall y,\ x + y^2 < 1$

10. Determine which constants satisfy the following sentential functions:
   (a) $\exists y,\ x = y^2$
   (b) $\exists y,\ x \cdot y = 1$

Notes

[1]By "arithmetic" we shall here understand that part of mathematics which is concerned with the investigation of the general properties of numbers, relations between numbers and operations on numbers. In place of the word "arithmetic" the term "algebra" is frequently used, particularly in high-school mathematics. We have given preference to the term "arithmetic" because, in higher mathematics, the term "algebra" is reserved for the much more special theory of algebraic equations. (In recent years the term "algebra" has obtained a wider meaning, which is, however, still different from that of "arithmetic".) – The term "number" will here always be used with that meaning which is normally attached to the term "real number" in mathematics; that is to say, it will cover integers and fractions, rational and irrational, positive and negative numbers, but not imaginary or complex numbers.

[2]Variables were already used in ancient times by Greek mathematicians and logicians, – though only in special circumstances and in rare cases. At the beginning of the 17th century, mainly under the influence of the work of the French mathematician F. Vieta (1540-1603), people began to work systematically with variables and to employ them consistently in mathematical considerations. Only at the end of the 19th century, however, due to the introduction of the notion of a quantifier, was the role of variables in scientific language and especially in the formulation of mathematical theorems fully recognized; this was largely the merit of the outstanding American logician and philosopher Ch. S. Peirce (1839-1914).

Chapter 2

On The Sentential Calculus

2.1 The Old Logic and the New Logic

The constants with which we have to deal in every day scientific theory may be divided into two large groups. The first group consists of terms which are specific for a given theory. In the case of arithmetic, for instance, they are terms denoting either individual numbers or whole classes of numbers, relations between numbers, operations on numbers, etc. The constants which we used in Section 1.1 belong to this first group, among others. On the other hand, there are terms of a much more general character occurring in most of the statements of arithmetic, terms which are met constantly both in considerations of every day life and in every possible field of science, and which represent an indispensable means for conveying human thoughts and for carrying out inferences in any field. Such words as "*not*", "*and*", "*or*", "*is*", "*every*", "*some*", and many others belong here. Theirs is a special discipline, namely *logic*, considered the basis for all the other sciences, whose concern it is to establish the precise meanings of such terms and to lay down the most general laws in which these terms are involved.

Logic developed into an independent science long ago, earlier even than arithmetic and geometry. And yet it has only been recently – after a long period of almost complete stagnation – that this discipline has begun an intensive development, in the course of which it has undergone a complete transformation with the effect of assuming a character similar to that of the mathematical disciplines. In this new form, it is known as *symbolic logic*. The new logic surpasses the old logic in many respects, not only because of the solidity of its foundations and the perfection of the methods employed in its development, but mainly on account of the wealth of concepts and theorems that have been established. Fundamentally, the old traditional logic forms only a fragment of the new, a fragment moreover which, from the point of view of the requirements

Logic, New New Edition.
Karl Frinkle.

of other sciences, and of mathematics in particular, is entirely insignificant. This text is aimed at the new logic, and we will not focus much of our attention on the realm of traditional logic.[1]

2.2 Negations, Conjunctions, and Disjunctions

Among the terms of a logical character there is a small distinguished group, consisting of such words as "*not*", "*and*", "*or*", "*if* ..., *then* ...". All these words are well-known to us from everyday language, and serve to build up compound sentences from simpler ones. In grammar, they are counted among the so-called *sentential conjunctions*. If only for this reason, the presence of these terms does not represent a specific property of any particular science. To establish the meaning and usage of these terms is the task of most elementary and fundamental part of logic, which is called *sentential calculus*. We will now discuss the meaning of the most important terms of sentential calculus.[2]

Definition 2.1. With the help of the word "*not*" one forms the *negation* of any sentence.

Definition 2.2. Two sentences, of which the first is a negation of the second, are called *contradictory sentences*.

In sentential calculus, the word "*not*" is put in front of the whole sentence, whereas in everyday language it is customary to place it with the verb; or should it be desirable to have it at the beginning of the sentence, it must be replaced by the phrase "*it is not the cast that*". Thus, for example, the negation of the sentence:

1 *is a positive integer*

reads as follows:

1 *is not a positive integer,*

or else:

it is not the case that 1 *is a positive integer.*

When we speak the negation of a sentence aloud, we normally intend to express the idea that the sentence is actually false. If the sentence was false, then its negation is true.

Definition 2.3. The joining of two or more sentences by the word "*and*" results in a compound sentence called a *conjunction* (or *logical product*).

If, for instance, the sentences:

2 *is a positive integer*

and

$$2 < 3$$

are joined in this way, we obtain the conjunction:

$$2 \textit{ is a positive integer and } 2 < 3.$$

The stating of the conjunction of two sentences is tantamount to stating that both sentences of which the conjunction is formed are true. If this is actually the case, then the conjunction is true, but if at least one of its members is false, then the whole conjunction is false.

Definition 2.4. By joining of two or more sentences by the word "*or*", we obtain the *disjunction* (or *logical sum*) of the sentences.

The word "*or*" in everyday language, possesses at least two different meanings. Taken in the so-called *non-exclusive* meaning, the disjunction of two sentences merely expresses that at least one of these sentences is true, without saying anything as to whether or not both sentences may be true. Taken in another meaning, the so-called *exclusive* one, the disjunction of two sentences asserts that one of the sentences is true but that the other is false. Suppose we see the following notice put up in a bookstore:

Customers who are teachers or college students are entitled to a discount.

Here the word "*or*" is undoubtedly used in the first sense, since it is not intended to refuse reduction to a teacher who is at the same time a college student. If, on the other hand, a child has asked to be taken on a hike in the morning and to the theatre in the afternoon, and we reply:

No, we are going on a hike or we are going to the theatre.

Then our usage of the word "*or*" is obviously of the second kind since we intend to comply with only one of the two requests. In logic and mathematics, the word "*or*" is always used in the non-exclusive meaning. The disjunction is considered true if at least one of its members are true; and otherwise false. Thus, for instance, it may be asserted:

Every number is positive or less than 3*,*

although it is known that there are numbers which are both positive and less than 3. In order to avoid misunderstandings, it would be expedient, in everyday as well as in scientific language, to use the word "*or*" by itself only in the first meaning, and to replace it by the compound expression "*either* ... *or* ..." whenever the second meaning is intended.

Even if we confine ourselves to those cases in which the word "*or*" occurs in its first meaning, we find quite noticeable differences between the usages of it in everyday language and in logic. In common language, two sentences are joined by the word "*or*" only when they are in some way connected in form and content. The same applies, though perhaps to a lesser degree, to the usage of the word "*and*". The nature of this connection is not quite clear, and a detailed analysis and description of it would meet with considerable difficulties. At any rate, anybody unfamiliar with the language of contemporary logic would presumably be little inclined to consider such a phrase as:

$$2 \cdot 2 = 5 \;\; \textit{or Durant is a city in Oklahoma.} \tag{2.1}$$

as a meaningful expression, and even less so to accept it as a true sentence. Moreover, the usage of the word "*or*" in everyday English is influenced by certain factors of a psychological character. Usually we affirm a disjunction of two sentences only if we believe that one of them is true but wonder which one. If, for example, we look upon a lawn in normal light, it would not enter our mind to say that the lawn is green or blue, since we are able to affirm something simpler and, at the same time, stronger, namely that the lawn is green. Sometimes even, we take the utterance of a disjunction as an admission by the speaker that they do not know which of the members of the disjunction is true. And if we later arrive at the conviction that he knew at the time that one – and, specifically, which – of the members was false, we are inclined to look upon the whole disjunction as a false sentence, even should the other member be undoubtedly true. Let us imagine, for instance, that a friend of ours, upon being asked when they are leaving town, answers that he is going to do so today, tomorrow, or the day after. Should we then later ascertain that, at that time, our friend had already decided to leave the same day, we shall probably get the impression that they were deliberately misleading us.

The creators of contemporary logic, when introducing the word "*or*" into their considerations, desired to simplify its meaning and to render the latter clearer and independent of all psychological factors, especially of the presence or absence of knowledge. Consequently, they extended the use of the word "*or*", and decided to consider the disjunction of any two sentences as a meaningful whole, even should no connection between their contents or forms exist; and they also decided to make the truth of a disjunction – like that of a negation or a conjunction – dependent only and exclusively upon the truth of its members. Therefore, the disjunction previously stated is meaningful from the viewpoint of contemporary logic. In fact, it is a true sentence. In a similar manner, the friend who was leaving town was telling the truth if we are operating under the premise of contemporary logic, which is independent of our opinion of their intentions.

2.3 Conditional Sentences and Implications in Material Meaning

Definition 2.5. Two sentences joined by the words "*if* ..., *then* ..." create the compound sentence known as the *conditional sentence*, or *implication*.

Definition 2.6. The subordinate clause to which the word "*if*" is prefixed in a conditional sentence is called the *antecedent*.

Definition 2.7. The principal clause introduced by the word "*then*" in a conditional sentence is called the *consequent*.

By asserting an implication one asserts that it does not occur that the antecedent is true and the consequent is false. An implication is thus true in any one of the following three cases:

(i) both antecedent and consequent are true,
(ii) the antecedent is false and the consequent is true,
(iii) both antecedent and consequent are false.

Only in the fourth case, when the antecedent is true and the consequent is false, is the whole implication false.

One of the most important results of the definition of the implication is that if one accepts that an implication is true, and at the same time accepts its antecedent as true, one must conclude that the consequent by itself is also true. This idea will be put to use in the *Rule of Detachment* (also known as Modus Ponens) which will be discussed later. Similarly, if one accepts an implication as true and rejects its consequent as false, one must also reject its antecedent as false.

Similar to the case of the disjunction, considerable differences between the usages of implication in logic and everyday language manifest themselves. Again, in ordinary language, we tend to join two sentences in an implication only when there is some connection between their forms and contents. This connection is hard to characterize in a general way, and only sometimes is its nature relatively clear. We often associate with this connection the conviction that the consequent follows necessarily from the antecedent, that is to say, that if we assume the antecedent to be true we are compelled to assume the consequent, too, to be true (and that possibly we can even deduce the consequent from the antecedent on the basis of some general laws which we might not always be able to quote explicitly). Here again, an additional psychological factor manifests itself; usually we formulate and assert an implication only if we have no exact knowledge as to whether or not the antecedent and consequent are true. Otherwise the use of an implication seems unnatural and its sense and truth may raise some doubt.

The following example may serve as an illustration. Let us consider the sentence:

Every metal is malleable.

One can convert this to an implication:

$$\text{If } x \text{ is a metal, then } x \text{ is malleable.} \tag{2.2}$$

If we believe in the truth of this universal law, we then must believe in all of its particular cases. If we replace "x" by names of arbitrary materials such as iron, clay, or wood, it turns out that all sentences obtained in this way satisfy the conditions given above for a true implication, i.e. it never happens that the antecedent is true while the consequent is false. Clearly there is a close connection between the antecedent and the consequent. For instance, "*iron is a metal*" is a true sentence, and we can thus deduce that "*iron is malleable*" using the law given in Sentence (2.2).

Nevertheless, some of the sentences discussed just now seem artificial and doubtful from the point of view of common language. No doubt is raised by the universal implication given above, or by any of its particular cases obtained by replacing "x" by the name of a material of which we do not know whether it is a metal or whether it is malleable. But if we replace "x" by "*iron*", we are confronted with a case in which the antecedent and consequent are undoubtedly true; and we shall then prefer to use, instead of an implication, an expression such as:

Since iron is a metal, it is malleable.

Similarly, if for "x" we substitute "*clay*", we obtain an implication with a false antecedent and a true consequent, and we shall be inclined to replace it by the expression:

Although clay is not a metal, it is malleable.

And finally, the replacement of "x" by "*wood*" results in an implication with a false antecedent and a false consequent; if in this case, we want to retain the form of an implication, we should have to alter the grammatical form of the verbs:

If wood were a metal, it would be malleable.

The logicians, with due regard for the needs of scientific languages, adopted the same procedure with respect to the phrase "*if* ..., *then* ..." as they had done in the case of the word "*or*". They decided to simplify and clarify the meaning of this phrase, and to free it from psychological factors. For this purpose, they extended the usage of this phrase, considering an implication as a meaningful sentence even if no connection whatsoever exists between its two members, and they made the truth or falsity of an implication dependent exclusively upon the truth or falsity of the antecedent and consequent. To characterize this situation briefly, we say that contemporary logic uses *implication in material meaning*, or simply *material implications*; this is opposed to the usage of *implication in formal meaning* or *formal implication*, in which case the presence of a certain formal connection between the antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication. The concept of formal implication is not, perhaps, quite clear, but, at any rate, it is narrower than that of material implication; every meaningful and true formal implication is at the same time a meaningful and true material implication, but not vice versa.

In order to illustrate the foregoing remarks, let us consider the following four sentences:

If $2 \cdot 2 = 4$, *then Durant is a city in Oklahoma.*
If $2 \cdot 2 = 5$, *then Durant is a city in Oklahoma.*
If $2 \cdot 2 = 4$, *then Durant is not a city in Oklahoma.*
If $2 \cdot 2 = 5$, *then Durant is not a city in Oklahoma.*

In everyday language, these sentences would hardly be considered as meaningful, and even less as true. From the point of view of mathematical logic, on the other hand, they are all meaningful, the third sentence being false, while the remaining three are true. Thereby, it is, of course, not asserted that sentences like these are particularly relevant from any viewpoint whatever, or that we apply them as premisses in our arguments.

It would be a mistake to think that the difference between everyday language and the language of logic, which has been brought to light here, is of an absolute character, and that the rules, outlined above, of the usage of the words "*if* ..., *then* ..." in common language admit no exceptions. Actually, the usage of these words fluctuates more or less, and if we look around, we can find cases in which this usage does not comply with our rules. Let us imagine that a friend of ours is confronted with a very difficult problem and that we do not believe that he will ever solve it. We can then express our disbelief by saying:

If you solve this problem, I shall eat my hat.

The tendency of this utterance is quite clear. We affirm here an implication whose consequence is undoubtedly false; therefore, since we affirm the truth of the whole implication, we thereby, at the same time, affirm the falsity of the antecedent; that is to say, we express our conviction that our friend will fail to solve the problem in which he is interested. But it is also quite clear that the antecedent and the consequent of our implication are in no way connected, so that we have a typical case of a material and not of a formal implication.

The divergency in the usage of the phrase "*if* ..., *then* ..." in ordinary language and mathematical logic has been at the root of lengthy and even passionate discussions. It has been objected that logicians, on account of their employment of the material implication, arrived at paradoxes and even plain nonsense.[3] This has resulted in an outcry for a reform of logic to the effect of bringing about a far-reaching rapprochement between logic and ordinary language regarding the use of implication.

It would be hard to grant that these criticisms are well-founded. There is no phrase in ordinary language that has a precisely determined sense. It would scarcely be possible to find two people who would use every word with exactly the same meaning, and even in the language of a single person the meaning of the same word varies from one period of their life to another. Moreover, the meaning of words of everyday language is usually very complicated; it depends not only on the external form of the word but also on the circumstances in which it is uttered and sometimes even on psychological factors. If a scientist wants to introduce a concept from everyday life into a science and to establish general laws concerning this concept, they must always make its content clearer, more precise and simpler, and free from inessential attributes; it does not matter here whether they are a logician concerned with the phrase "*if* ..., *then* ..." or, for instance, a materials scientist establishing the exact meaning of the word "*metal*". In whatever way a scientist realizes their task, the usage of the term as it is established by them will deviate more or less from the practice of everyday language. If, however, they state explicitly in what meaning they decide to use the term, and consistently act in conformity to this decision, nobody will be in a position to object that his procedure leads to nonsensical results.

2.4 The Use of Implication in Mathematics

The phrase "*if* ..., *then* ..." belongs to those expressions of logic which are used most frequently in other sciences and, especially, in mathematics. Mathematical theorems, particularly those of a universal character, tend to have the form of implications; the antecedent is called in mathematics the *hypothesis* and the consequent is called the *conclusion*.

As a simple example of a theorem of arithmetic, having the form of an implication, we may quote the following sentence:

$$\textit{If } x \textit{ is a positive number, then } 2x \textit{ is a positive number.} \tag{2.3}$$

Here, "x *is a positive number*" is the hypothesis, while "$2x$ *is a positive number*" is the conclusion.

Apart from this, so to speak, classical form of mathematical theorems, there are, occasionally, different formulations, in which hypothesis and conclusion are connected in some other way by the phrase "*if* ..., *then* ...". The theorem just mentioned, for instance, can be paraphrased in any of the following forms:

From: x is a positive number, it follows: $2x$ is a positive number.

The hypothesis: x is a positive number, implies (or has as a consequence) the conclusion: $2x$ is a positive number.

The condition: x is a positive number, is sufficient for $2x$ to be a positive number.

For $2x$ to be a positive number it is sufficient that x be a positive number.

The condition: $2x$ is a positive number, is necessary for x to be a positive number.

For x to be a positive number it is necessary that $2x$ be a positive number.

Therefore, instead of asserting a conditional sentence, one might usually just as well say that the hypothesis *implies* the conclusion or *has as its consequence*, or that it is a *sufficient condition* for the conclusion; or one can express it by saying that the conclusion *follows* from the hypothesis, or that it is a *necessary condition* for the latter. A logician may raise various objections against some of the formulations given above, but they are general use in mathematics.

The objections which might be raised here concern those of the above formulations in which any of the words "*hypothesis*", "*conclusion*", "*consequence*", "*follows*", "*implies*" occur. In order to understand the essential points in these objections, we observe first that those formulations differ in content from the ones originally given. While in the original formulation we talk only about numbers, properties of numbers, operations upon numbers and so on – hence, about objects with which mathematics is concerned–, in the formulations now under discussion we talk about hypotheses, conclusions, conditions, that is about sentences or sentential functions occurring in mathematics. It might be noted on this occasion that, in general, people do not distinguish clearly enough the terms which denote objects dealt with in a given science from those which denote various kinds of expressions occurring within it. This can be observed, in particular, in the domain of mathematics, especially on the elementary level. Presumably only few are aware of the fact that such terms as "*equation*", "*inequality*", "*polynomial*", or "*algebraic fraction*", which are met at every turn in textbooks of elementary algebra, do not, strictly speaking, belong to the domain of mathematics or logic, since they do not denote objects considered in this domain; equations and inequalities are certain special sentential functions, while polynomial and algebraic fractions – especially as they are treated in elementary textbooks – are particular instances designatory functions (c.f. Section 1.2). The confusion on this point is brought about by the fact that terms of this kind are frequently used in the formulation of mathematical theorems. This has become a very common usage, and perhaps it is not worth our while to put up a stand against it, since it does not present any particular danger; but it might be worth our while to get to recognize that, for every theorem formulated with the help of such terms, there is another formulation, logically more correct, in which those terms do not occur at all. For instance the theorem:

$$\textit{The equation: } x^2 + ax + b = 0 \textit{ has at most two roots.} \tag{2.4}$$

can be expressed in a more correct manner as follows:

$$\textit{There are at most two numbers } x \textit{ such that: } x^2 + ax + b = 0. \tag{2.5}$$

Returning to the questionable formulations of an implication, we must emphasize one still more important point. In these formulations we assert that one sentence, namely the antecedent of the implication has another – the consequent of the implication – as a consequence, or that the second follows from the first. But ordinarily when we express ourselves in this way, we have in mind that the assumption that the first sentence is true leads us, so to speak, necessarily to the same assumption concerning the second sentence (and that possibly we are even able to derive the second sentence from the first). As we already know from Section 2.3, however, the meaning of an implication, as it was established in contemporary logic, does not depend on whether its consequent has any such connection with its antecedent. Anyone shocked by the fact that expression (2.1) is considered in logic as meaningful and even true sentences will find it still harder to reconcile himself with such a transformation of this phrase as:

The hypothesis that $2 \cdot 2 = 4$ *has as a consequence that Durant is a city in Oklahoma.*

We see, thus, that the manners discussed here of formulation or transforming a conditional sentence lead to paradoxical sounding statements and make yet more profound the discrepancies between common language and mathematical logic. From the purely logical point of view we can obviously avoid all objections raised here by stating explicitly once and for all that, in using the formulations in question, we shall disregard their usual meaning and attribute to them exactly the same content as to the ordinary conditional sentence. But this would be inconvenient in another respect; for there are situations – though not in logic itself, but in a field closely related to it, namely the methodology of deductive sciences (cf. Chapter 6) – in which we talk about sentences and the relation of consequence between them, and in which we use such terms as "*implies*" and "*follows*" in a different meaning more closely akin to the ordinary one. It would, therefore, be better to avoid those formulations altogether, all the more since we have several formulations at our disposal which are not open to any of these objections.

2.5 Equivalence of Sentences

We shall consider one more expression from the field of sentential calculus. It is one which is comparatively rarely met in everyday language, namely the phrase "*if, and only, if*".

Definition 2.8. If any two sentences are joined by the phrase *if, and only, if*, the result is a sentence called a *biconditional sentence*, or an *equivalence*.

The two sentences connected in this way are referred to as the *left* and *right side of the equivalence*. By asserting the equivalence of two sentences, it is intended to exclude the possibility that one is true and the other false.

Definition 2.9. Two statements are said to be *equivalent* if they always have the same truth value; that is, if the left and right sides of the biconditional sentence formed by the two statements are either both true or both false.

The sense of an equivalence can also be characterized in still another way, but first we need a new definition related to the conditional sentence:

Definition 2.10. If, in a conditional sentence, we interchange the antecedent and consequent, we obtain a new sentence which, in its relation to the original sentence, is called the *converse sentence*.

Let us take, for instance, as the original sentence the implication (2.3). The converse of this sentence will then be:

$$\textit{If } 2x \textit{ is a positive number, then } x \textit{ is a positive number.} \tag{2.6}$$

As is shown by this example, it occurs that the converse of a true sentence is true. In order to see, on the other hand, that this is not a general rule, it is sufficient to consider the sentence:

$$\textit{If } x \textit{ is a positive number, then } x^2 \textit{ is a positive number.} \tag{2.7}$$

whose converse sentence is given by

$$\textit{If } x^2 \textit{ is a positive number, then } x \textit{ is a positive number.} \tag{2.8}$$

Note that even though the conditional sentence (2.7) is true, its converse sentence (2.8) is false. For instance, any negative value of x will be positive upon being squared, thus making the hypothesis of the converse statement true, while its conclusion is clearly false.

To construct an equivalence, we simply require that that two conditional sentences, of which one is the converse of the other, are both true, and then join them with the phrase "*if, and only, if*". As an example, we can take the two sentences (2.3) and (2.6) and create the single compound sentence:

$$x \textit{ is a positive number if and only if, } 2x \textit{ is a positive number.} \tag{2.9}$$

or similarly,

$$2x \textit{ is a positive number if and only if, } x \textit{ is a positive number.} \tag{2.10}$$

There are, incidentally, still a few more possible formulations which may serve to express the same idea (similar to that of the conditional statements):

From: x is a positive number, it follows: $2x$ is a positive number, and conversely.

The conditions that x is a positive number and that $2x$ is a positive number are equivalent with each other.

The condition that x is a positive number is both necessary and sufficient for $2x$ to be a positive number.

For x to be a positive number it is necessary and sufficient that $2x$ be a positive number.

Instead of joining two sentences by the phrase "*if, and only if*", it is therefore, in general, also possible to say that the *relation of consequence* holds between these two sentences *in both directions*, or that the two sentences are *equivalent*, or finally, that each of the two sentences represents a *necessary and sufficient condition* for the other.

2.6 Formulations of Definitions and Its Rules

The phrase "*if, and only if*" is very frequently used in laying down *definitions*, that is, conventions stipulating what meaning is to be attributed to an expression which thus far has not occurred in a certain discipline, and which may not be immediately comprehensible. Imagine, for instance, that in arithmetic the symbol "$\leq$" has not as yet been employed but that one wants to introduce it now into the considerations. For this purpose it is necessary to define the symbol first, that is, to explain exactly its meaning in terms of expressions which are already known and whose meanings are beyond doubt. To achieve this, we lay down the following definition, assuming that "$>$" belongs to the symbols already known:

Definition 2.11. We say that $x \leq y$ if, and only if, it is not the case that $x > y$.

The definition formulated above states the equivalence of the two sentential functions

$$x \leq y$$

and

It is not the case that $x > y$;

it may be said, therefore, that it permits the transformation of the formula "$x \leq y$" into an equivalent expression which no longer contains the symbol "$\leq$" but is formulated entirely in terms already comprehensible to us. The same holds for any formula obtained from "$x \leq y$" by replacing "x" and "y" by arbitrary symbols or expressions designating numbers. For example, the formula:

$$3 + 2 \leq 5$$

is equivalent with the sentence:

It is not the case that $3 + 2 > 5$;

since the latter is a true assertion, so is the former. Similarly, the formula:

$$4 \leq 2 + 1$$

is equivalent with the sentence:

It is not the case that $4 > 2 + 1$;

both being false assertions. This remark applies also to more complicated sentences and sentential functions; by transforming, for instance, the sentence:

If $x \leq y$ *and* $y \leq z$, *then* $x \leq z$,

we obtain:

If it is not the case that $x > y$ *and if it is not the case that* $y > z$, *then it is not the case that* $x > z$.

In short, by virtue of the definition given above, we are in a position to transform any simple or compound sentence containing the symbol "$\leq$" into an equivalent one no longer containing it; in other words, to translate it into a language in which the symbol "$\leq$" does not occur. And it is this very fact which constitutes the role which definitions play within the mathematical disciplines.

If a definition is to fulfil its proper task well, certain precautionary measures have to be observed in its formulation. To this effect, special rules are laid down, the *Rules of Definition*, which specify how definitions should be constructed correctly. Since we shall not here go into an exact formulation of these rules, it may merely be remarked that, on their basis, every definition may assume the form of an equivalence; the first member of that equivalence, the *definiendum*, should be a short, grammatically simple sentential function containing the constant to be defined; the second member, the *definiens*, may be a sentential function of an arbitrary structure, containing, however, only constants whose meaning either is immediately obvious or has been explained previously. In this text, when we are specifically defining a new term, we shall usually express it in the following way:

$$\textit{definiendum} \overset{def}{\longleftrightarrow} \textit{definiens} \tag{2.11}$$

In particular, the constant to be defined, or any expression previously defined with its help, must not occur in the definiens; otherwise the definition is incorrect, it contains an error known as a *vicious circle in the definition.* In a similar fashion, one encounters a *vicious circle in the proof* if the argument meant to establish a certain theorem is based upon the theorem itself, or upon some theorem previously proved with its help. In order to emphasize the conventional character of a definition and to distinguish it from other statements which have the form of an equivalence, it is expedient to prefix it by words such as "*we say that*". It is easy to verify that the above definition of the symbol "$\leq$" satisfies these conditions; it has the definiendum

$$x \leq y;$$

whereas the definiens reads:

$$\textit{It is not the case that } x > y.$$

It is worth noticing that mathematicians, in laying down definitions, prefer the words "*if*" or "*in case that*" to the phrase "*if, and only if*". If for example, they had to formulate the definition of the symbol "$\leq$', they would, presumably, give it the following form:

Definition 2.12. We say that $x \leq y$ if it is not the case that $x > y$.

It looks as if such definition 2.12 merely states that the definiendum follows from the definiens without emphasizing that the relation of consequence also holds in the opposite direction, and this fails to express the equivalence of definiendum and definiens. But what we have here is a tacit convention to the effect that "*if*" or "*in the case that*", if used to join definiendum and definiens, are to mean the same as the phrase "*if, and only if*" ordinarily does. It may be added that the form of an equivalence is not he only in which definitions may be laid down.

2.7 The Laws of Sentential Calculus

After having come to the end of our discussion of the most important expressions of sentential calculus, we shall now try to clarify the character of the laws of this calculus.

Let us consider the sentence

If 1 *is a positive number and* $1 < 2$*, then* 1 *is a positive number.* (2.12)

This sentence is obviously true, it contains exclusively constants belonging to the field of logic and arithmetic, and yet the idea of listing this sentence as a special theorem in a textbook of mathematics would not occur to anybody. If one reflects why this is so, one comes to the conclusion that this sentence is completely uninteresting from the standpoint of arithmetic; it fails to enrich in any way our knowledge about numbers, its truth does not at all depend upon the content of the arithmetical terms occurring in it, but merely upon the sense of the words "*and*", "*if*", "*then*". In order to make sure that this is so, let us replace in Sentence (2.12) the components: 1 *is a positive number*, and $1 < 2$ by any other sentences from an arbitrary field; the result is a series of sentences, each of which, like the original sentence, is true; for example:

If the given figure is a rhombus and if the same figure is a rectangle, then the given figure is a rhombus.

It today is Sunday and the sun is shining, then today is Sunday.

Definition 2.13. A *sentential variable* is a variable which represents a whole sentence, and is not the designation of a number or any other object.

We shall replace in Sentence (2.12) the antecedent by "p", and the consequent by "q"; in this manner, we arrive at the sentential function:

Law 2.1 (The Law of And, Breaking Apart)**.** *If* p *and* q*, then* p*.*

This sentential function has the property that only true sentences are obtained if arbitrary sentences are substituted in "p" and "q". This observation may be given in the form of a universal statement:

Law 2.2. *For any* p *and* q*, if* p *and* q*, then* p*.*

We have here obtained a first example of a law of sentential calculus, which is historically referred to as the *Law of Simplification for logical multiplication*, but in an attempt to describe it a little more easily we also will call it *The Law of And* (*Breaking Apart*). We shall see more laws of simplification shortly, and will need to discern between them. The sentences involving the rhombus and days of the week are both considered to be *instances of this law* – just as, for instance, the formula:

$$2 \cdot 3 = 3 \cdot 2$$

is merely a special instance of the universal arithmetical theorem:

Theorem 2.1. *For arbitrary numbers x and y, $x \cdot y = y \cdot x$.*

In a similar way, other laws of sentential calculus can be obtained. We give here a few examples of such laws; in their formulation we omit the universal quantifier "*for any p, q, ...*" – in accordance with the usage mentioned in Section 1.3, which becomes almost a rule throughout sentential calculus.

Law 2.3 (The Law of Identity)**.** *If p, then p.*

Law 2.4 (The Law of Or, Joining Together)**.** *If p, then q or p.*

Law 2.5 (The Law of Hypothetical Syllogism)**.** *If p implies q and q implies r, then p implies r.*

Law 2.6 (The Law of Biconditional)**.** *If p implies q and q implies p, then p if, and only if, q.*

Just as the arithmetical theorems of a universal character state something about the properties of arbitrary numbers, the laws of sentential calculus assert something, so one may say, about the properties of arbitrary sentences. The fact that in these laws only such variables occur as stand for quite arbitrary sentences is characteristic of sentential calculus and decisive for its great generality and the scope of its applicability.

2.8 Symbolism and Truth Tables

There exists a certain simple and general method, called the *Method of Truth Tables*, which enables us, in any particular case, to recognize whether a given sentence from the domain of the sentential calculus is true, and whether, therefore, it can be counted among the laws of this calculus.[4]

In describing this method, it is convenient to apply a special symbolism. We shall replace the logical connectives with symbols:

expression	symbol
not	$\sim$
and	$\wedge$
or	$\vee$
if, ... then ...	$\rightarrow$
... if, and only if ...,	$\leftrightarrow$

Table 2.1: Symbols for logical connectives

The first of these symbols is a unary operator, which means it is to be placed in front of the expression whose negation one wants to obtain. The remaining symbols are binary operators and must be placed between two expressions.

From one or more simpler expressions we are, in this way, led to a more complicated expression; and if we want to use the latter for the construction of further still more complicated expressions, we enclose it in parentheses.

With the help of variables, parentheses, and the connective symbols listed above (and sometimes also additional constants well defined and of a similar nature dependent upon the circumstances), we are able to write down all sentences and sentential functions belonging to the domain of sentential calculus. Apart from the individual sentential variables the simplest sentential functions are the expressions:

$$\sim p, \quad p \wedge q, \quad p \vee q, \quad p \rightarrow q, \quad p \leftrightarrow q$$

As an example of a compound sentential function let us consider the expression

$$(p \vee q) \rightarrow (p \wedge r), \tag{2.13}$$

which we read, translating symbols into common language:

If p or q, then p and r.

A still more complicated expression is Law 2.5, the Law of Hypothetical Syllogism, which now assumes the form:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \wedge r), \tag{2.14}$$

We can easily make sure that every sentential function occurring in our calculus is a so-called *truth function*. This means to say that the truth or falsity of any sentence obtained from that function by substituting whole sentences for variables depends exclusively upon the truth or falsehood of the sentences which have been substituted. As for the simplest sentential functions "$\sim p$", "$p \wedge q$" and so on, this follows immediately from the remarks made in Sections 2.2, 2.3, and 2.5 concerning the meaning attributed in logic to the words "*not*", "*and*", and so on. But the same applies, likewise, to compound functions. Let us consider, for instance, the Sentence (2.13). A sentence obtained from it by substitution is an implication, and, therefore, its truth depends on the truth of its antecedent and consequent only; the truth of its antecedent, which is a disjunction obtained from "$p \vee q$" depends only on the truth of the sentences substituted for "p" and "q", and similarly the truth of the consequent depends only on the truth of the sentences substituted for "p" and "r". Thus, finally the truth of the whole sentence obtained from the Sentence (2.13) depends exclusively on the truth of the sentences substituted for "p", "q", and "r".

In order to see quite exactly how the truth or falsity of a sentence obtained by substitution from a given sentential function depends upon the truth or falsity of the sentences substituted for variables, we construct a *truth table*. We shall begin by giving such a table for the function "$\sim p$":

In Table 2.2, the "T" and "F" stand for "true sentence" and "false sentence", respectively. These are the only two possibilities for a single sentential

p	$\sim p$
T	F
F	T

Table 2.2: Fundamental truth table for negation

function "p". The negation of said sentence, as discussed in Section 2.2, simply switches the truth value of "p". The remaining four logical connectives require two sentential functions, which we will denote as "p" and "q", and there will be four distinct possible pairings of truth values for these two sentential functions, and thus four rows in our truth table.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
F	T	F	T	T	F
T	F	F	T	F	F
F	F	F	F	T	T

Table 2.3: Fundamental truth table for the binary connectives

In Table 2.3, we find, in the second line below the headings "p", "q", and "$p \rightarrow q$", the letters "F", "T", and "T", respectively. We can therefore concede that a sentence obtained from the implication "$p \rightarrow q$" is true if we substitute any false sentence for "p" and any true sentence for "q"; this obviously is entirely consistent with the remarks made in Section 2.3. It should be noted that the variables "p" and "q" could be replaced by any other variable. Tables 2.2 and 2.3 are called *fundamental truth tables*, as they are the basis for computing the truth tables of all other sentential functions based on the connectives "$\sim$", "$\wedge$", "$\vee$", "$\rightarrow$", and "$\leftrightarrow$". Truth tables derived from these so-called fundamental truth tables are referred to as *derivative truth tables*. We will next derive the truth table for Sentence (2.13).

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \rightarrow (p \wedge r)$
T	T	T	T	T	T
F	T	T	T	F	F
T	F	T	T	T	T
F	F	T	F	F	T
T	T	F	T	F	F
F	T	F	T	F	F
T	F	F	T	F	F
F	F	F	F	F	T

Table 2.4: Derivative truth table for Sentence (2.13)

In order to explain the construction of Table 2.4, let us concentrate, say, on its fifth horizontal line below the headings. We substitute true sentences for "p" and "q" and a false sentence for "r". According to the Table 2.3, we then obtain from "$p \vee q$" a true sentence and from "$p \wedge r$" a false sentence. From the whole sentential function (2.13) we obtain then an implication with a true antecedent and a false consequent; hence, again with the help of Table 2.3, in which we think of "p" and "q" being momentarily replaced by "$p \vee q$" and "$p \wedge r$" respectively, we conclude that this implication is a false sentence.

The horizontal lines of a table that consist of the symbols "T" and "F" are called *rows* of the table, and the vertical lines are called *columns*. Each row, or, rather, that part of each row which is on the left of the vertical bar represents a certain substitution of true or false sentences for the variables. When constructing the matrix of a given function, we take care to exhaust all possible ways in which a combination of symbols "T" and "F" can be correlated to the variables; and, of course, we never write in a table two rows which do not differ either in number or in the order of the symbols "T" and "F". It can be seen very easily that the number of rows in a table depends in a simple way on the number of different variables occurring in the function; if a function contains, n variables, then there will be 2^n rows. The simplest way to ensure that all possible combinations of "T" and "F" are exhausted in the truth table is to alternate "T" and "F" by powers of 2. For instance, Table 2.4 has three variables, hence 2^3 rows. The first columns, corresponding to the variable "p" alternates between "T" and "F" every 2^0 entries, while the second variable, "q", alternates every 2^1 entries. The last variable column, "r" alternates every $2^{3-1} = 2^2$ entries. In a similar fashion, if one were to construct a truth table involving four simple variables, there would be 2^4 rows in the table, and to compute all possible combinations of "T" and "F", we would alternate the values of "T" and "F" every 2^0, 2^1, 2^2 and 2^3 entries for columns 1, 2, 3 and 4, respectively. As for the number of columns, it is equal to the number of partial sentential functions of different form contained in the given function (where the whole function is also counted among its partial functions).

We are now in a position to say how it may be decided whether or not a sentence of sentential calculus is true. As we know, in sentential calculus, there is no external difference between sentences and sentential functions; the only difference consisting in the fact that the expressions considered to be sentences are always completed mentally by the universal quantifier. In order to recognize whether the given sentence is true, we treat it, for the time being, as a sentential function, and construct the truth table for it. If, in the last column, no symbol "F" occurs, then every sentence obtainable from the function in question by substitution will be true, and therefore our original universal sentence (obtained from the sentential function by mentally prefixing the universal quantifier) is also true. If, however, the last column contains at least one symbol "F", our sentence is false. This leads to the following definition.

Definition 2.14. A sentential function whose derivative truth table's last column consists only of the truth value of "T" is said to be a *tautology*, or is referred to as a *tautological sentence*. This type of statement is always true regardless of the truth value of any individual variable in the sentential function form of the sentence.

Thus, for instance, we have seen that in the Truth Table 2.4 for Sentence (2.13), the symbol "F" occurs four times in the last column. If, therefore, we considered the expression prefixed by the words "*for any p, q, and r*", we would have a false sentence. We may conclude that Sentence (2.13) is not a tautology. On the other hand, it can be easily verified with the help of the method of truth tables that all the laws of sentential calculus stated in Section 2.7, that is, the laws of simplification, identity, and so on, are true sentences. The table for Law 2.1, is as follows:

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

Table 2.5: Derivative truth table for Law 2.1

For more complicated expressions, such as the following two:

$$[\,(p \to q) \wedge (p \to r)\,] \to [\,p \to (q \vee r)\,], \tag{2.15}$$

$$[\,(p \wedge q) \to r\,] \to [\,(p \to r) \vee (q \to r)\,]. \tag{2.16}$$

it is more convenient and space saving to simply write the truth values of each operation (unary or binary) below the operation itself, remembering the order of operations. To compute the truth table of the first of the above expressions, one must first compute the truth values of the two expressions

$$(p \to q), (p \to r),$$

and then their conjunction

$$(p \to q) \wedge (p \to r).$$

This takes care of the antecedent of the overall conditional sentence. Next, the truth table for the disjunction

$$q \vee r$$

is computed, and then

$$p \to (q \vee r).$$

Lastly, the overall conjunction is evaluated. The final truth value for the sentence (2.15) is given in the fourth column to the right of the vertical line in Table 2.6 below. Note that every entry in this "final column"(in bold below) has a value of "T".

p	q	r	$[(p$	$\to$	$q)$	$\wedge$	$(p$	$\to$	$r)]$	$\to$	$[p$	$\to$	$(q$	$\vee$	$r)]$
T	T	T		T		T		T		**T**		T		T	
F	T	T		T		F		F		**T**		T		T	
T	F	T		F		F		T		**T**		T		T	
F	F	T		F		F		F		**T**		F		F	
T	T	F		T		T		T		**T**		T		T	
F	T	F		T		T		T		**T**		T		T	
T	F	F		T		T		T		**T**		T		F	
F	F	F		T		T		T		**T**		T		F	

Table 2.6: Derivative truth table for Sentence (2.15)

We now list a number of important laws of sentential calculus whose truth can be ascertained in a similar way.

Law 2.7 (The Law of Contradiction)**.** $\sim [p \wedge (\sim p)]$.

Law 2.8 (The Law of Excluded Middle)**.** $p \vee (\sim p)$.

Law 2.9 (The Law of And Tautology)**.** $(p \wedge p) \leftrightarrow p$.

Law 2.10 (The Law of Or Tautology)**.** $(p \vee p) \leftrightarrow p$.

Law 2.11 (The Law of Commutative And)**.** $(p \wedge q) \leftrightarrow (q \wedge p)$.

Law 2.12 (The Law of Commutative Or)**.** $(p \vee q) \leftrightarrow (q \vee p)$.

Law 2.13 (The Law of Associative And)**.** $[(p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$.

Law 2.14 (The Law of Associative Or)**.** $[(p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$.

It can easily be seen how obscure the meanings of Laws 2.13 and 2.14 become if we try to express them in ordinary language. This exhibits very clearly the value of logical symbolism as a precise instrument for expressing more complicated thoughts.

It occurs that the method of truth tables leads us to accept sentences as true whose truth seemed to be far from obvious before the application of this method. Here are some examples of sentences of this kind:

$$\begin{aligned}
&p \to (q \to p), \\
&(\sim p) \to (p \to q), \\
&(p \to q) \vee (q \to p).
\end{aligned}$$

That these sentences are not immediately obvious is due mainly to the fact that they are a manifestation of the specific usage of implication characteristic of modern logic, namely, the usage of implication in material meaning.

These sentences assume an especially paradoxical character if, when reading them in words of common language, the implications are replaced by phrases containing "*implies*" or "*follows*", that is, if we give them, for instance, the following form:

If p is true, then p follows from any q (in other words: *a true sentence follows from every sentence*);

If p is false, then p implies any q (in other words: *a false sentence implies any sentence*);

For any p and q, either p implies q or q implies p (in other words: *at least one of any two sentences implies the other*).

In this formulation, these statements have frequently been the cause of misunderstanding and superfluous discussions. This confirms entirely the remarks of Section 2.4.

2.9 Application of Laws of Sentential Calculus in Inference

Almost all reasonings in any scientific domain are based explicitly upon laws of sentential calculus; we shall try to explain by means of an example in what way this happens.

Given a sentence having the form of an implication, we can, apart from its converse of which we have already spoken in Section 2.5, form two further sentences.

Definition 2.15. The *inverse* sentence to a conditional sentence is obtained by replacing both the antecedent and consequent of the conditional sentence by their negations.

Definition 2.16. The *contrapostive* sentence to a conditional sentence is the result of interchanging the antecedent and the consequent in the inverse sentence.

From the definition of the contrapositive, we see that the contrapositive is the converse of the inverse sentence, and also the inverse of the converse sentence.

Definition 2.17. The converse, the inverse, and the contrapositive sentences, together with the original sentence, are referred to as *conjugate sentences*.

As an illustration, we will revisit Sentence (2.3):

If x is a positive number, then $2x$ is a positive number.

The three conjugate sentence, the converse, inverse, and contrapositive are given as follows:

If $2x$ is a positive number, then x is a positive number.

If x is not a positive number, then $2x$ is not a positive number.

If $2x$ is not a positive number, then x is not a positive number.

In this particular example, all the conjugate sentences obtained from a true sentence turn out to likewise be true. But this is not at all so in general; in order to see that it is quite possible that not only the converse sentence (as has already been mentioned in Section 2.5) but also the inverse sentence may be false, although the original sentence is true, it is sufficient to replace "$2x$" by "x^2" in the above sentences.

Thus, it is seen that from the validity of an implication nothing definite can be inferred about the validity of the converse or the inverse sentences. The situation is quite different in the case of the fourth conjugate sentence, the contrapositive. Whenever an implication is true; the same applies to the corresponding contrapositive sentences. This fact may be confirmed by numerous examples, and it finds its expression in a general law of sentential calculus, known as the *Law of Contraposition.*

In order to be able to formulate this with precision, we observe that every implication may be given the schematic form $p \to q$, the converse, inverse, and contrapositive sentences. The following table gives the symbolic representations of each conjugate sentence.

conditional	$p \to q$
converse	$q \to p$
inverse	$\sim p \to \sim q$
contrapositive	$\sim q \to \sim p$

Table 2.7: Symbolic form of the four conjugate sentences.

Law 2.15 (The Law of Contraposition)**.** *Any conditional sentence implies its corresponding contrapositive sentence, and may be formulated symbolically as:*

$$(p \to q) \to (\sim q \to \sim p)$$

We now want to show how, with the help of Law 2.15, we can, from a statement having the form of an implication – for instance, from Sentence (2.3)

– derive its contrapositive statement. Upon substituting for p: "*x is a positive number*", and for q: "*2x is a positive number*" into Law 2.15, we obtain an instance of the Law given as follows:

From: if $2x$ ...

Now if we compare the instance of the Law of Contraposition we just created, as well as Sentence (2.3), we note that Sentence (2.17) has Sentence (2.3) as its hypothesis. Since the whole implication as well as the hypothesis have been acknowledged as true, the conclusion of Sentence (2.17) must also be true; but this is just the contrapositive of the question:

If $2x$ *is a positive number, then* x *is a positive number.* (2.18)

Anyone knowing the Law of Contraposition can, in this way, recognize the contrapositive sentence as true, provided that the original implication is known to be true. In a similar fashion, the inverse sentence is contrapositive with respect to the converse of the original sentence (that is to say, the inverse sentence can be obtained from the converse sentence by replacing antecedent and consequent by their negations and then interchanging them); for this reason, if the converse of the given sentence has been proved, the inverse sentence may likewise be considered valid. If, therefore, one has succeeded in proving two sentences – the original and its converse – a special proof for our two remaining conjugate sentences is superfluous.

It may be mentioned that several variants of the Law of Contraposition are known, one of them is the converse of Law 2.15.

Law 2.16 (The Law of Converse Contraposition)**.** *The* Law of Contraposition*, according to which any conditional sentence implies the corresponding contrapositive sentence, may be formulated equivalently in terms of its contrapositive form:*

$$(\sim q \to \sim p) \to (p \to q)$$

This law makes it possible to derive the original sentence from the contrapositive, and the inverse from the converse sentence.

2.10 Rules of Inference, Complete Proofs

We shall now consider in more detail the mechanism itself of the proof by means of which Sentence (2.18) had been demonstrated in the preceding section. Besides the rules of definition, of which we have already spoken, we have other rules of a somewhat similar character, namely the *Rules of Inference*. These rules, which must not be mistaken for logical laws, amount to directions as to how sentences already known as true may be transformed so as to yield new true sentences. In the derivation of the contrapositive sentence (2.18), two rules of have been made us of: the *Rule of Substitution* and the *Rule of Detachment* (also known as the *Modus Ponens Rule*).

Rule of Substitution. *If a sentence of a universal character, that has already been accepted as true, contains sentential variables, and if these variables are replaced by other sentential variables or by sentential functions or sentences – always substituting equal expressions for equal variables throughout –, then the sentence obtained in this way may be recognized as true.*

It was by applying this very rule that we obtained Sentence (2.17) from Law 2.15. We call this *an instance* of Law 2.15 (or more generally, an instance of a specific sentence). It should be noted that the *Rule of Substitution* may also be applied to other kinds of variables, for example, to the variables "x", "y", ... designating numbers; in place of these variables, any symbols or expressions denoting numbers may be substituted.

The formulation for the *Rule of Substitution* just given is not quite complete. This rule refers to such sentences as are composed of a universal quantifier and a sentential function, the latter containing variables bound by the universal quantifier. When one wants to apply the *Rule of Substitution*, one omits for the quantifier and substitutes for the variables previously bound by the quantifier either other variables or whole expressions (e.g. sentential functions or sentences for the variables "p", "q", "r", ..., and expressions denoting numbers for the variables "x", "y", "z", ...); any other bound variables which may occur in the sentential function remain unaltered, and one sees to it that no variables of the same form as these occur in the substituted expressions; if necessary a universal quantifier is set in front of the expression obtained in this way in order to turn it into a sentence. For instance, by applying the *Rule of Substitution* to the sentence:

For any number x there is a number y such that $x + y = 5$

the following sentence can be obtained:

There is a number y such that $3 + y = 5$,

but also the sentence:

For any number z there is a number y such that $z^2 + y = 5$.

Thus, in this case, one substitutes only for "x" and leaves "y" unaltered. We must not, however, substitute for "x" any expression containing "y"; for, although our original sentence was true, we might in this way arrive at a false sentence. For instance, by substituting "$3 - y$", we obtain:

There is a number y such that $(3 - y) + y = 5$.

Substitution creates new sentences by replacing variables and expressions for other variables and expressions. However, the form of the sentence remains the same. The *Rule of Detachment* allows us to create entirely new sentences in regards to structure.

Rule of Detachment. *If two sentences are accepted as true, of which one has the form of an implication while the other is the antecedent of this implication, then that sentence may also be recognized as true, which forms the consequent of the implication.* (*We* detach *thus, so to speak, the antecedent from the whole implication.*).

By means of this rule, Sentence (2.18) was derived from Sentences (2.17) and (2.3). In the proof of the Sentence (2.18) as carried out previously, each step consisted of applying a rule of inference to sentences which were previously accepted or recognized as true. A proof of this kind is called *complete*. A little more precisely a complete proof may also be characterized as follows. It consists in the construction of a chain of sentences with these properties: the initial members are sentences which were already previously accepted as true; every subsequent member is obtained from preceding ones by applying a rule of inference; and finally the last member is the sentenced to be proved.

We will reproduce the proof previously done, but in a shorthand notation which we will employ throughout the rest of the text. Pay attention to the proof and the description of a complete proof from the previous paragraph.

(1)	If x is a positive number, then $2x$ is a positive number.	Assumption
(2)	From: if x is a positive number, then $2x$ is a positive number, it follows that: if $2x$ is not a positive number, then x is not a positive number.	Instance of the Law of Contraposition, $p : x$ is a positive number $q : 2x$ is a positive number
(3)	If $2x$ is not a positive number, then x is not a positive number.	Rule of Detachment, (1) and (2)

It should be observed what an extremely elementary form all mathematical reasonings assume, due to the knowledge and application of the laws of logic and the rules of inference; complicated mental processes are entirely reducible to such simple activities as the attentive observation of statements previously accepted as true, the perception of structural, purely external, connections among these statements, and the execution of mechanical transformations as prescribed by the rules of inference. It is obvious that, in view of such a procedure, the possibility of committing mistakes in a proof is reduced to a minimum. To illustrate, we can take the complete proof given above, and reduce it to its symbolic logic form, and thus the resulting proof can be used on any argument of this form.

(1)	$p \to q$	Assumption
(2)	$(p \to q) \to (\sim q \to \sim p)$	Law of Contraposition
(3)	$\sim q \to \sim p$	Rule of Detachment on (1) and (2)

Exercises

1. Differentiate in the following two sentences between the specifically mathematical expressions and those belonging to the domain of logic:

(a) *For any numbers x and y, if $x > 0$ and $y < 0$, then there is a number z such that $z < 0$ and $x = y \cdot z$.*

(b) *For any points A and B there is a point C which lies between A and B and is the same distance from A as from B.*

2. Consider the following two sentential functions: $x < 3$, and $x > 3$. (a) Form the negation of the conjunction of the sentential functions, and (b)determine which numbers satisfy the compound sentence.

3. Consider the following conditional sentences:

(a) *If today is Monday, then tomorrow is Tuesday.*
(b) *If today is Monday, then tomorrow is Saturday.*
(c) *If today is Monday, then the* 25*th of December is Christmas day.*

Which of the above implications are true and which are false from the point of view of mathematical logic? In which cases does the question of meaningfulness and of truth or falsity raise any doubt from the standpoint of ordinary logic? Direct special attention to sentence (b) and examine the question of its truth as dependent on the day of the week on which it was uttered.

4. Put the following theorem into the form of an ordinary conditional sentence:

The condition: x is divisible by 3*, is necessary for x to be divisible by* 6*.*

5. Give alternative formulations for the following sentence:

For a quadrangle to be a parallelogram it is necessary and sufficient that the midpoint of both diagonals are the points of intersection between the diagonals.

6. Determine which of the following sentences are true:

(a) *The fact that a quadrangle is a square implies that all its angles are right angles, and conversely.*

(b) *For x to be divisible by* 8 *it is necessary and sufficient that x be divisible both by* 4 *and by* 2.

7. Assuming the terms "*natural numbers*" and "*product*" (or "*quotient*", respectively) to be known already, construct the definition of the term "*divisible*", giving it the form of an equivalence:

We say that x is divisible by y if, and only if, ...

8. Translate the following symbolic expressions into ordinary language:

(a) $[(\sim p) \to p] \to p$
(b) $[(\sim p) \vee q] \leftrightarrow (p \to q)$
(c) $[\sim (p \vee q)] \leftrightarrow (p \to q)$
(d) $(\sim p) \vee [q \leftrightarrow (p \to q)]$

Direct special attention to the difficulty in distinguishing in ordinary language the three last expressions.

9. Construct truth tables for all sentential functions given in Exercise 8 and state which sentences are tautologies and which are not.

10. Formulate the following expressions in logical symbolism:

(a) *If p or not q, then it is not the case that p or q.*
(b) *If p implies that q implies r, then p and q together imply r.*

11. Verify by the method of truth tables that the following sentences are tautologies:

(a) $[\sim (\sim p)] \leftrightarrow p$
(b) $[\sim (p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$
(c) $[\sim (p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$
(d) $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
(e) $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$

Sentence (a) is the *Law of Double Negation*, sentences (b) and (c) are called *De Morgan's Laws*[5], and sentences (d) and (e) are the *Distributive Laws*.

12. For the following sentence, state the three corresponding conjugate sentences:

The fact that x is a positive number implies $-x$ is a negative number.

13. For the sentence in Exercise 12, determine which of the conjugate sentences are true.

14. Consider the following two sentences:
(a) *from: if p, then q, it follows that: if q, then p.*
(b) *from: if p, then q, it follows that: if not p, then not q.*

Suppose these sentences were logical laws, would it be possible to apply them in mathematical proofs in an analogous way to the Law of Contraposition (cf. Sections 2.9 and 2.10)? What conjugate sentences would it be possible to derive from a given asserted implication? Consequently, can our supposition be maintained that the sentences (a) and (b) are true?

15. Consider the following two statements:
The fact that yesterday was Monday implies that today is Tuesday.
The fact that today is Tuesday implies that tomorrow will be Wednesday.

What statement may be deduced from them in accordance with Law 2.5, the Law of Hypothetical Syllogism?

16. Carry out the complete proof of the statement obtained in the preceding exercise; use the statements and the Law of Hypothetical Syllogism mentioned there, and apply – in addition to the Rule of Detachment – the following rule of inference:

Rule of And, Joining Together. *If any two sentences are accepted as true, then their conjunction may be recognized as true.*

Notes

[1]Logic was created by Aristotle, the Greek thinker of the 4th century B.C. (384-322); his logical writings are collected in the work Organon. As the creator of mathematical logic we have to look upon the great German philosopher and mathematician of the 17th century G.W. Leibniz (1646-1716). However, the logical works of Leibniz failed to have a great influence upon the further development of logical investigations; there was even a period in which they sank into oblivion. A continuous development of mathematical logic began only towards the middle of the 19th century, namely at the time when the logical system of the English mathematician G. Boole was published (1815-1864; principal work: *An Investigation of the Laws of Thought*, London 1854). So far the new logic has found its most perfect expression in the epochal work of the great contemporary English logicians A. N. Whitehead and B. Russell: *Principia Mathematica* (Cambridge, 1910-1913).

[2]The historically first system of sentential calculus is contained in the work *Begriffsschrift* (Halle 1879) of the German logician G. Frege (1848-1925) who, without doubt, was the greatest logician of the 19th century. The eminent contemporary Polish logician and historian of logic J. Lukasiewicz succeeded in giving sentential calculus a particularly simple and precise form and caused extensive investigations concerning this calculus.

[3]It is interesting to notice that the beginning of this discussion dates back to antiquity. It was the Greek philosopher Philo of Megara (in the 4th century B.C.) who presumably was the first in history of logic to propagate the usage of material implication; this was in opposition to the views of his master, Diodorus Cronus, who proposed to use implication in a narrower sense, rather related to what is called here the formal meaning. Somewhat later (in the 3rd century B.C.) – and probably under the influence of Philo – various possible conceptions of implication were discussed by the Greek philosophers and logicians of the Stoic School (in whose writings the first beginnings of sentential calculus are to be found).

[4]This method originates with Pierce (who has already been cited at an earlier occasion; cf. endnote 2 of Chapter 1).

[5]These laws were given by A. De Morgan (1806-1878), an eminent English Logician.

Chapter 3

On The Theory of Identity

3.1 Logical Concepts Outside Sentential Calculus

Sentential calculus, to which the preceding chapter was devoted, forms merely a part of logic. It constitutes undoubtedly the most fundamental part, – at least inasmuch as one makes use of the terms and laws of this calculus in the definition of terms and the formulation and demonstration of logical laws that do not belong to sentential calculus. Sentential calculus in itself, however, does not form a sufficient basis for the foundation of other sciences and, in particular, not of mathematics; various concepts from other parts of logic are constantly encountered in mathematical definitions, theorems, and proofs. Some of them will be discussed in the present chapter, and in the following two chapters.

Among the logical concepts not belonging to sentential calculus, the concept of *identity* or *equality* is probably the one which has the greatest importance. It occurs in phrases such as:

x is identical with y,
x is the same as y,
x equals y.

To all three of these expressions the same meaning is ascribed; for the sake of brevity, they will be replaced by the symbolic expression:

$$x = y.$$

Instead of writing:

x is not identical with y,

or instead of writing:

Logic, New New Edition.
Karl Frinkle.

x is different from y,

we employ the formula:

$$x \neq y.$$

The general laws involving the above expressions constitute a part of logic which may be called the *Theory of Identity*.

3.2 Fundamental Laws of the Theory of Identity

Among the logical laws concerning the concept of identity the most fundamental is the following:

Law 3.1 (The Law of Identity). *$x = y$ if, and only if, x has every property y has, and y has every property that x has.*

We could also say more simply:

$x = y$ if, and only if, x and y have every property in common.

Other and perhaps more apparent, though less correct, formulations of the same law are known, for instance:

$x = y$ if, and only if, everything that may be said about any one of the objects x or y may also be said about the other.

The Law of Identity was first stated by Leibniz[1] (although in somewhat different terms) and hence may be called *Leibniz's Law.* It has the form of an equivalence, and enables us to replace the formula:

$$x = y,$$

which is the left side of the equivalence (*definiendum* cf. equation (2.11)), by its right side (*definiens*), that is by an expression no longer containing the symbol of identity. With respect to its form this law may, therefore, be considered as the definition of the symbol "=", and so it was considered by Leibniz himself. (Of course, Lebniz's Law here as a definition would make sense only if the meaning of the symbol "=" seemed to us less evident than that of the expressions on the right side of the law, such as "*x has every property which y has*", cf. Section 2.6).

As a consequence of Leibniz's Law we have the following rule which is of great importance:

Rule of Replacement. *If, in a certain context, a formula having the form of an equation, e.g.:*

$$x = y,$$

has been assumed or proved, then it is permissible to replace, in any formula or sentence occurring in this context, the left side of the equation by its right side, e.g. "x" by "y", and conversely.

It is understood that, should "x" occur at several places in a formula, it may at some places be left unchanged and at others be replaced by "y"; there is, thus, an essential difference between the Rule of Substitution of Section 2.10, which does not permit a partial replacement of one symbol by another, and the Rule of Replacement now under consideration.

From Leibniz's Law we can derive a number of other laws belonging to the Theory of Identity that are frequently applied in various considerations and especially in mathematical proofs. The most important among these will be listed here, together with their proofs, in order to exhibit by way of concrete examples that there is no essential difference between reasonings in the field of logic, and those in the field of mathematics.

Law 3.2 (Law of Reflexive Identity)**.** *Every object is equal to itself: $x = x$.*

Proof. First we replace "y" with "x" in Leibniz's Law to get:

(1) $x = x \longleftrightarrow x$ has every property which x has $\wedge$ x has every property which x has.

Note that the right-hand side is of the form $p \wedge p$, which, by the Law of And Tautology is equivalent to p. Thus, using the Rule of Substitution on the Law of And Tautology with p: x has every property which x has, gives:

(2) x has every property which x has $\wedge$ x has every property which x has $\longleftrightarrow x$ has every property which x has.

Note that the left-hand side of (2) is the right side of (1), so we can use the Rule of Substitution and substitute (2) into (1) to get:

(3) $x = x \longleftrightarrow x$ has every property which x has.

The right hand side of (3) is a true statement, thus we can consider the sentence by itself due to the biconditional identity: (True $\leftrightarrow p) \leftrightarrow p$

(4) x has every property which x has.

Now we substitute (3) into (4) to get

(5) $x = x$.

Sentence (5) is, of course, the sentence we were trying to prove, and therefore the proof of Law 3.2 is complete. □

Law 3.3 (Law of Symmetric Identity)**.** *If $x = y$, then $y = x$.*

Proof. Upon replacing "x" by "y" and "y" by "x" in Leibniz's Law gives:

(1) $y = x \leftrightarrow y$ has every property which x has $\wedge\, x$ has every property which y has.

If we could switch the order of the conjunction on the right hand side of (1), we would have the right hand side of Leibniz Law. So we can use an instance of the Law of Commutative And (Law 2.11), setting p: y has every property that x has, and q: x has every property that y has, to get the statement:

(2) y has every property which x has $\wedge\, x$ has every property which y has $\leftrightarrow$ x has every property which y has $\wedge\, y$ has every property which x has.

Next we substitute (via the Rule of Substitution) the right side of (2) into the right side of (1) since the right side of (1) is the left side of (2). This yields:

(3) $y = x \leftrightarrow x$ has every property which y has $\wedge\, y$ has every property which x has.

Note now that the right side of (3) is the right side of Leibniz Law, thus substituting the left hand side of Leibniz Law into the right hand side of (3) gives

(4) $y = x \leftrightarrow x = y$

What we currently have is a statement which appears to be even more than we are looking for. We only need to prove a conditional sentence, where we have a biconditional sentence in (4). Thus, we can use an instance of:

(5) $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

with p: $x = y$ and q: $y = x$: to get

(6) $(x = y \leftrightarrow y = x) \rightarrow (x = y \rightarrow y = x)$

We can now use the Rule of Detachment on (4) and (6) to get

(7) $x = y \rightarrow y = x$

Thus, our proof is complete. □

We next prove a slightly more complicated Law, which requires a somewhat different approach when attempting to prove.

Law 3.4 (Law of Transitive Identity)**.** *If $x = y$ and $y = z$, then $x = z$.*

Proof. We start with a new approach here, we will assume the antecedent (hypothesis) of this implication, and using the rules of logic, arrive at the consequent (conclusion) of the implication. So our first step is to assume:

(1) $x = y \wedge y = z$

Next, we will break up the conjunction to get each simple sentence by itself. To accomplish this, we will use The Law of And, Breaking Apart (Law 2.1). The instance will be with $p : x = y$, and $q : y = x$.

(2) $(x = y \wedge y = z) \to x = y$

We can use a modified version of the Law of And, Breaking Apart, with $(p \wedge q) \to q$ (where the consequent is q instead of p) to get a similar statement to (2) but with the second sentence instead:

(3) $(x = y \wedge y = z) \to y = z$

Next, we use the Rule of Detachment on (1) and (2), and also on (1) and (3). Although painful to take these tedious steps, they are very simple and straightforward. These two applications of the Rule of Detachment yield:

(4) $x = y$

(5) $y = z$

Next, we write down Leibniz's Law, and an instance of Leibniz's Law with "x" replaced with "y" and "y" with "z".

(6) $x = y \leftrightarrow x$ has every property which y has $\wedge\, y$ has every property which x has.

(7) $y = z \leftrightarrow y$ has every property which z has $\wedge\, z$ has every property which y has.

Substituting (6) and (7) into (4) and (5) yields the right hand sides of (6) and (7):

(8) x has every property which y has $\wedge\, y$ has every property which x has.

(9) y has every property which z has $\wedge\, z$ has every property which y has.

Using the Law of And, Breaking Apart on (8) and (9), we can get the left sentence in each conjunction:

(10) x has every property which y has.

(11) y has every property which z has.

Next, we use The Rule of And, Joining Together (from the end of the Exercises of Chapter 2) on (10) and (11) to get:

(12) x has every property which y has $\wedge\, y$ has every property which z has.

From (12), we may conclude, by the rule of replacement, that:

(13) x has every property which z has.

Similarly, if we use the Law of And, Breaking Apart on (8) and (9), we can get the right sentence in each conjunction, and then the Rule of And, Joining Together, to join these two together to get:

(14) z has every property which x has.

Using the Rule of And on (13) and (14) gives:

(15) x has every property which z has $\wedge\, z$ has every property which x has.

Note that (15) is the right hand side of Leibniz's Law with "y" replaced by "z":

(16) $x = z \leftrightarrow x$ has every property which z has $\wedge\, z$ has every property which x has.

Substituting (16) into (15) gives:

(17) $x = z$.

Finally we have reached the consequent of our law, and the proof is complete. If you pay attention to all of the steps we have taken, it may appear that we have taken way too many steps to complete the proof. In future chapters, we will abbreviate the explanation of these steps. But for now, we will not take anything for granted, the proof is in the details. □

Law 3.5. *If $x = z$ and $y = z$, then $x = y$.*

Proof. We assume the antecedent of this implication:

(1) $x = z \wedge y = z$

Similar to the proof in the previous law, we need to break up the conjunction into its two simple sentences. For this, we will use The Law of And, Breaking Apart (Law 2.1) and the Rule of Detachment. This gives:

(2) $x = z$

(3) $y = z$

We will now use an instance of Law of Symmetric Identity, replacing "x" with "y" and "y" with "z":

(4) $y = z \to z = y$

Since (3) is the antecedent of (4), we can use the Rule of Detachment to get:

(5) $z = y$

Next, we use The Rule of And, Joining Together with (2) and (5):

(6) $x = z \wedge z = y$

Sentence (6) looks very similar to the Law of Transitive Identity, except with "y" and "z" switched, so we perform that substitution in the law to get:

(7) $(x = z \wedge z = y) \to x = y$

One more application of the Rule of Detachment, this time with (6) and (7) gives the consequent of (7):

(8) $x = y$

□

Although the law we just proved seemed just as complicated as the Law of Transitive Identity, once we have proven both the Law of Symmetric Identity and the Law of Transitive Identity, we can use them to prove this law. Instead of twenty steps or more to prove this, it took only eight. As a rule, the more laws one has proven, the more possible approaches one can use to complete the proof of any new theorem arising from them.

3.3 Identity of Objects, Their Designations, and Quotation Marks

Although the meaning of such expressions as:

$$x = y \text{ or } x \neq y$$

seem to be evident, these expressions are sometimes misunderstood. It seems obvious, for instance, that the formula:

$$3 = 2 + 1, \tag{3.1}$$

is a true assertion, and yet some people are somewhat doubtful as to its truth. In their opinion, this formula appears to state that the symbols "3" and "$2+1$"

are identical, which is obviously false since these symbols have entirely different shapes, and, therefore, it is not true that everything that may be said about one of these symbols may be said about the other as required by Leibniz's Law. For instance, the first symbol is a single sign, while the second is not.

In order to avoid doubts of this kind, it is well to make clear to oneself a very general and important principle upon which the useful employability of any language is dependent. According to this principle, whenever, in a sentence, we wish to say something about a certain object, we have to use, in this sentence, not the object itself but its name or designation.

The application of this principle gives no cause for doubt as long as the object talked about is not a word, a symbol or, more generally, an expression of language. Let us imagine, for example, that we have a small blue stone in front of us, and that we state the following sentence:

This stone is blue.

To none, presumably, would it occur in this case, to replace in this sentence the words "*this stone*" which together constitute the designation of the object by the object itself, that is to say, to plot out the words and replace them with an actual blue stone. For in doing so, we would arrive at a whole consisting in part of a stone and in part of words, and thus, at something which would not be a linguistic expression, and far less a true sentence.

This principle is, however, frequently violated if the object talked about happens to be a word or symbol. And yet, the application of the principle is indispensable also in this case; for, otherwise, we would arrive at a whole which, though being a linguistic expression, would fail to express the thought intended by us, and very often might even be a meaningless aggregate of words. Let us consider, for example, the following two words:

fast, *Mary.*

Clearly, the first consists of four letters, and the second is a proper name. But let us imagine that we would express these thoughts, which are quite correct, in the following manner:

Fast consists of four letters. (3.2)

Mary is a proper name. (3.3)

we would then, in talking about words, be using the words themselves and not their names. And if we examine Sentences (3.2) and (3.3) more closely, we must admit that the first is not a sentence at all since the subject can only be a noun and not an adverb, while the second might be considered a meaningful sentence, but, at any rate, a false one since no woman is a proper name.

In order to avoid these difficulties, we might assume that the words "*fast*" and "*Mary*" occur in such contexts as (3.2) and (3.3) in a meaning distinct from the usual one, and that they here function as their own names. In generalization of this viewpoint, we should have to admit that any word may, at times, function as its own name; to use the terminology of medieval logic, we may say that, in a case like this, the word is used in *suppositio materialis*, as opposed to its use in *suppositio formalis*, that is, in its ordinary meaning. As a consequence, every word of common or scientific language would possess at least two different meanings, and one would not have to look far for examples of situations in which serious doubts might arise as to which meaning was intended. With this consequence we do not wish to reconcile ourselves, and therefore we will make it a rule that every expression should differ (at least in writing) from its name.

The problem arises as to how we can set about to form names of words and expressions. There are various devices to this effect. The simplest one among them is based on the convention of forming the name of an expression by placing it between quotation marks. On the basis of this agreement, the thoughts tentatively expressed in Sentences (3.2) and (3.3) can now be stated correctly and without ambiguity as follows:

$$\text{“\textit{Fast}” \textit{consists of four letters.}} \tag{3.4}$$

$$\text{“\textit{Mary}” \textit{is a proper name.}} \tag{3.5}$$

In light of these remarks all possible doubts as to the meaning and the truth of such formulas as (3.1) are dispelled. Equation 3.1 contains symbols designating certain numbers, but it does not contain the names of any such symbols. Therefore, this formula states something about numbers and not about symbols designating numbers; the numbers 3 and $2 + 1$ are obviously equal, so that the formula is a true assertion. We may, admittedly, replace this formula by an equivalent sentence which is about symbols, namely, we may say that the symbols “3” and “$2 + 1$” designate the same number. But this by no means implies that the symbols themselves are identical; for it is well known that the same objects – and, in particular, the same number – can have many different designations. The symbols “3” and “$2 + 1$” are, no doubt, different, and this face can be expressed in the form of a new formula

$$\text{“3”} \neq \text{“}2 + 1\text{”} \tag{3.6}$$

which, of course, in no way contradicts Sentence (3.1).[2]

3.4 Equality in Arithmetic and Geometry

We here consider the notion of arithmetical equality among numbers consistently as a special case of the general concept of logical identity. It must be added, however, that there are mathematicians who – as opposed to the standpoint adopted here – do not identify the symbol “=” occurring in arithmetic with the symbol of logical identity; they do not consider equal numbers to be necessarily identical, and therefore look upon the notion of equality among numbers as a specifically arithmetical concept. In this connection, those mathematicians reject Leibniz's Law in its general form, and merely recognize some of its consequences which are of a less general character and count them among the specifically mathematical theorems. Among these consequences there are Laws 3.2 through 3.5 of Section 3.2, as well as theorems to the effect that whenever $x = y$ and x satisfies some formula built up of arithmetical symbols only, then y satisfies the same formula; thus for instance, the theorem:

Theorem 3.1. *If $x = y$, and $x < z$, then $y < z$.*

In our opinion, this point of view can claim no particular theoretical advantages, while, in practice, it entails considerable complications in the presentation of the system of arithmetic. For one rejects here the general rule which allows us – on the assumption that an equation holds – to replace everywhere the left side of the equation by its right side; since, however, such a replacement is indispensable in various arguments, it becomes necessary to give a special proof that this replacement is permissible in each particular case in which it is applied.

To illustrate this situation by example, let us consider any system of equations in two variables, for instance:

$$x = y^2$$
$$x^2 + y^2 = 2x - 3y + 18.$$

If one wants to solve this system of equations by means of the method of substitution, one has to form a new system of equations obtained by leaving the first equation unchanged and replacing in the second equation "x" by "y" throughout. And the question arises whether this transformation is permissible, that is, whether the new system is equivalent to the old. The is undoubtedly in the affirmative, no matter what conception of the notion of equality among numbers is adopted. But if the symbol "=" is understood to designate logical identity, and if Lebniz's Law is assumed, the answer is obvious; the assumption

$$x = y^2$$

permits us to replace "x" everywhere by "y^2", and vice versa. Otherwise it would first be necessary to give reasons for the affirmative answer, and although this justification would not meet with any essential difficulties, it would at any rate be rather long and tedious.

As to the notion of equality in geometry, the situation is entirely different. If two geometrical figures, such as two line segments, or two angles, or two polygons, are said to be equal or congruent, it is in general not intended to assert their identity. One merely wishes to state that the two figures have the same size and shape, in other words, that they would exactly cover one another if one were placed on top of the other. Thus, for example, a triangle is capable of having two, or even three, equal sides, and yet these sides are certainly not identical. There are also cases, on the other hand, in which it is not a question of the geometrical equality of two figures, but of their logical identity; for instance, in an isosceles triangle, the altitude upon the base and the median to the base are not only geometrically equal, but they are simply one and the same line segment. Therefore, in order to avoid any confusion, it would be recommendable consistently to avoid the term "equality" in all those cases where it is not a question of logical identity, and to speak of geometrically equal figures rather of congruent figures, replacing, as it is often done anyhow, the symbol "$=$" by a different one, such as "$\cong$".

3.5 Numerical Quantifiers

With the help of the concept of identity it is possible to give a precise meaning to certain phrases which, both in their context and their function, are closely related to the universal and existential quantifiers and are also counted among the operators, but which are of a more special character. They are expressions such as:

There is at least one, or *at most one*, or *exactly one object x such that…*

There are at least two, or *at most two*, or *exactly two objects x such that…*

and so on; and are referred to as *numerical quantifiers*. Apparently, specifically mathematical terms occur in these phrases, namely, the numerals "*one*", "*two*", and so on. A more exact analysis shows, however, that the content of those phrases (if considered as a whole) is of a purely logical nature. Thus, in the expression:

There is at least one object satisfying the given condition.

the words "*at least one*" may simply be replaced by the article "*a*" without altering the meaning. The expression:

There is at most one object satisfying the given condition.

means the same as:

For any x and y, if x satisfies the given condition and if y satisfies the given condition, then $x = y$.

The sentence:

There is exactly one object satisfying the given condition.

is equivalent with the conjunction of the two sentences just given:

There is exactly one object satisfying the given condition, and at the same time there is at most one object satisfying the given condition.

To the expression:

There is at least two objects satisfying the given condition.

we give the following meaning:

There are x and y, such that x and y satisfy the given condition and $x \neq y$.

It is, therefore, equivalent to the negation of:

There is at most one object satisfying the given condition.

Analogously we explain the meanings of other expressions of this category.

For the purpose of illustration, a few true sentences of arithmetic may be listed here in which numerical quantifiers appear:

There is exactly one number x, such that $x + 2 = 5$.

There are exactly two numbers y, such that $y^2 = 4$.

There are at least two numbers z, such that $z + 2 < 6$.

We introduced the symbols "$\forall$" and "$\exists$" in Section 1.4, and we are now in a position to write numerically quantified statements using the quantifiers. We already know that "*at least one*" is the $\exists$ quantifier, but what about "*exactly one*"? We could define this as "*at least one*" and "*at most one*", which still begs the question: How do we quantify "*at most one*"? Consider the following quantified statement, where $P(x)$ is defined to mean "*x has the property P*" (for some arbitrary property P):

$$\forall x, y \ \ ((P(x) \wedge P(y)) \rightarrow (y = x)) \tag{3.7}$$

To see that this is indeed the definition of "*at most one*", consider the following argument. If no x or y have the property P, then the antecedent of Sentence 3.7 is false, hence the whole statement is true. If we happen to have x and y which both have the property P, then the antecedent is true, and to make the whole statement true, the consequent, $x = y$ must also be true. Therefore, no more than one value can have the property P.

With just a slight modification to the definition above for "*at most one*", we arrive at the definition of "*exactly one*":

$$\exists x \ \forall y \ \ [\, P(x) \wedge \{\, P(y) \rightarrow (y = x) \,\}\,] \tag{3.8}$$

Here, the x is existentially quantified, thus giving the existence of at least one object with the property P. And if for any object y, y has the property P, then that y must be the object x from the existential quantifier. In a similar fashion, we can also define "*at most two*":

$$\forall x, y, z \; [\{ P(x) \wedge P(y) \wedge P(z) \} \rightarrow \{ (z = y) \vee (z = x) \vee (y = x) \}], \qquad (3.9)$$

and "*exactly two*":

$$\exists x, y \, \forall z \; [\{ P(x) \wedge P(y) \wedge \sim (x = y) \} \wedge \{ P(z) \rightarrow [(z = y) \vee (z = x)] \}] \quad (3.10)$$

That part of logic, in which general laws involving quantifiers are laid down, is known as the *Theory of Apparent Variables*, or the *Functional Calculus*, although it really ought to be called the *Calculus of Quantifiers*. Hitherto this theory has primarily concerned itself with universal and existential quantifiers, while the numerical quantifiers have been largely neglected.

Exercises

1. Prove Law 3.5 of Section 3.2, using exclusively Laws 3.3 and 3.4.

Hint: In Law 3.5 the formulas:

$$x = z \text{ and } y = z$$

are assumed valid by hypothesis. By virtue of Law 3.3, interchange the variables in the second of these formulas, and then apply Law 3.4.

2. Prove the following using exclusively Law 3.4 of Section 3.2:

Law 3.6. $(x = y \wedge y = z \wedge z = t) \rightarrow x = t$

3. Are the sentences true which are obtained by replacing in Laws 3.3 and 3.4 of Section 3.2 the symbol "=" by "≠" throughout?

4. On the basis of the convention of Section 3.3 concerning the use of quotation marks, determine which of the following expressions are true sentences:
   (a) 0 *is an integer.*
   (b) 0 *is a cipher having an oval shape.*
   (c) " 0" *is an integer.*
   (d) " 0" *is a cipher having an oval shape.*
   (e) $1.5 = \frac{3}{2}$
   (f) "1.5" = "$\frac{3}{2}$"
   (g) $2 + 2 \neq 5$
   (h) "$2 + 2$" ≠ "5"

5. In order to form the name of a word, we put it in quotation marks; in order to form the name of this name, we put, in turn, the name of this word in quotation marks, and thus the word itself in double quotation marks. Hence, of the following three expression:

John, “*John*”, ““*John*””,

the second is the name of the first, and the third is the name of the second. Substitute in turn the three expressions above for “x” in the following sentential functions, and determine which of the twelve sentences obtained are true:
(a) *x is a person.*
(b) *x is the name of a person.*
(c) *x is an expression.*
(d) *x is an expression containing quotation marks.*

6. Express each of the following quantified expressions using numerical quantifier notations similar to those of Sentences (3.8) and (3.9):
(a) *There are at least two objects satisfying the given condition.*
(b) *There are at most three objects satisfying the given condition.*

7. Determine which of the following sentences are true:
(a) *There is exactly one number x such that* $x + 3 = 7 - x$.
(b) *There is exactly one number x such that* $x + 3 = 7 + x$.
(c) *There are exactly two numbers x such that* $x^2 + 4 = 4x$.
(d) *There are at most two numbers y such that* $y + 5 < 11 - 2y$.

8. If, instead of considering all real numbers in Exercise 7, we consider only whole numbers (e.g. $\{0, 1, 2, \ldots\}$, does the truth value of any of the sentences change?

9. In the following sentential, determine which variables are free and which are bound. After doing so, answer the following question: Do numerical quantifiers bind variables?

There are exactly two numbers y such that $x = y^2$.

Notes

[1]Cf. endnote 1 of Chapter 1

[2]The convention concerning the use of quotation marks has been adhered to in this book pretty consistently. We deviate from it only in rare cases, by way of a concession to traditional usage. For instance, we state formulas and sentences without quotation marks, if they are printed, displayed in a special line, or of they occur in the formulation of mathematical or logical theorems; and we do not put quotation marks about expressions which are preceded by such phrases as “is called”, “is known as”, and so on. But other precautionary measures

are taken in these cases; the expression in question is often preceded by a colon and it is usually printed in a different style (like italics). It should be observed that, in every day language, quotation marks are used also in cases not covered by the above convention; and examples of this type can be found in this book, too.

Chapter 4

On The Theory of Classes

4.1 Classes and Their Elements

Apart from separate individual objects, which we shall also, for short, call individuals, logic is concerned with *classes* of objects; in everyday life as well as in mathematics, classes are more often referred to as *sets*.

Definition 4.1. A *class*, or *set*, is a collection of individuals (or elements), of which can usually be defined by specific list of properties.

Arithmetic, for instance, frequently deals with sets of numbers, and in geometry our interest attaches itself not so much to single points as to point sets (namely, to geometric configurations). Classes of individuals are called *Classes of the First Order*. Comparatively more rarely we also meet in our investigations with *Classes of the Second Order*, that is, classes which consist, not of individuals, but of classes of the first order. Sometimes, even *Classes of the Third Order*, *Fourth Order*, and so on, have to be dealt with. Here we shall be concerned almost exclusively with classes of the first order, and only exceptionally the second order; our considerations can, however, be applied with practically no changes to classes of any order.

In order to differentiate between individuals(elements) and classes (and also between classes of different orders), we employ as variables letters of different shape and belonging to different alphabets. It is customary to designate individual objects such as numbers, and classes of objects, by the small and capital letters of the English alphabet, respectively. In elementary geometry the opposite notation is the accepted one, capital letters designating points and small letter (of the English or Greek alphabets) designating point sets.

We can give examples of classes of first and second orders quite readily. For instance, the set of integers, usually denoted by the symbol $\mathbb{Z}$, is defined as:

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

Logic, New New Edition.
Karl Frinkle.

whereas the set of all prime numbers less than 10 is given by:

$$\{2, 3, 5, 7\}.$$

To illustrate a class of the second order, we will consider sets of numbers from the above set:

$$\{\{2\}, \{2, 3\}, \{5, 7\}, \{2, 3, 5\}, \{3, 5, 7\}\}$$

Note that the elements in the set given above are sets themselves. It is not too difficult to imagine what a class of the third order, or of the fourth order, would look like.

That part of logic in which the class concept and its general properties are examined is called the *Theory of Classes*; sometimes this theory is also treated as an independent mathematical discipline under the name of the *General Theory of Sets*, or *Set Theory*.[1]

Of fundamental character in the theory of classes are such phrases as:

The object x is an element (or a member) of the class K.

The element x belongs to the class K.

The class K contains the object x as an element (or a member).

We consider these expressions as having the same meaning and, for the sake of brevity, replace them by the formula:

$$x \in K.$$

Thus, the numbers -2, 0, and 3 are all elements of $\mathbb{Z}$, whereas the numbers $\frac{2}{3}$, 2.5, and π do not belong to the set $\mathbb{Z}$. Hence the formulas:

$$-2 \in \mathbb{Z}, \quad 0 \in \mathbb{Z}, \quad 3 \in \mathbb{Z},$$

are true, while the formulas:

$$\frac{2}{3} \in \mathbb{Z}, \quad 2.5 \in \mathbb{Z}, \quad \pi \in \mathbb{Z}$$

are false.

4.2 Classes and Sentential Functions with One Free Variable

We consider a sentential function with one free variable, for instance

$$x > 0 \tag{4.1}$$

If we prefix the words:

$$\textit{The set of all numbers } x \textit{ such that} \tag{4.2}$$

to sentential function (4.1), we obtain the expression:

$$\textit{The set of all numbers } x \textit{ such that } x > 0 \tag{4.3}$$

This expression designates a well-determined set, namely the set of all positive numbers; it is the set having the elements those, and only those, numbers which satisfy the given function. If we denote this set by the symbol "P", our function becomes equivalent to:

$$x \in P$$

This, of course, requires that the set P is well defined.

We may apply an analogous procedure to any other sentential function. In arithmetic, we can obtain in this way various sets of numbers, for instance the set of all negative numbers or the set of all numbers which are greater than 2 and less than 5 (that is which satisfy the function "$x > 2 \wedge x < 5$"). This procedure plays also an important role in geometry, especially in defining new kinds of geometry, especially in defining new kinds of geometrical configurations; the surface of a sphere is defined, for instance, as the set of all points of the space which have a definite distance from a given point. It is customary in geometry to replace the words "*the set of all points*" by "*the locus of points*".

We will now put the above remarks in a general form. It is assumed in logic that, to every sentential function containing just one free variable, say "x", there is exactly one corresponding class having as its elements those, and only those, objects x which satisfy the given function. We obtain a designation for that class by putting in front of the sentential function the following phrase, which belongs to the fundamental expressions of the theory of classes:

$$\textit{The class of all objects } x \textit{ such that} \tag{4.4}$$

If we denote further the class in question by a simple symbol, say "K", the formula

$$x \in K \tag{4.5}$$

will, for any x, be equivalent to the original sentential function.

Hence it is seen that any sentential function containing "x"as the only free variable can be transformed into an equivalent function of the form (4.5), where in place of "K" we have a constant denoting a class; one may, therefore consider the latter formula as the most general form of a sentential function with one free variable.

The phrases (4.2) and (4.5) are sometimes replaced by symbolic expressions; we can, for instance, agree to use the following symbol for this purpose:

$$\underset{x}{\mathrm{C}}$$

Let us now consider the following expression:

$$1 \textit{ belongs to the set of all number } x \textit{ such that } x > 0. \tag{4.6}$$

which can also be written in symbols exclusively as

$$1 \in \underset{x}{\mathrm{C}}(x > 0). \tag{4.7}$$

This expression is obviously a sentence, and even a true sentence; it expresses, in a more complicated form, the same thought as the simple formula:

$$1 > 0.$$

Consequently, this expression cannot contain any free variable, and the variable "x" occurring in it must be a bound variable. Since, on the other hand, we do not find in the above expression any quantifiers, we arrive at the conclusion that such phrases as (4.2) and (4.5) function like quantifiers, that is, they bind variables, and must, therefore, be counted among the operators (cf. Section 1.4).

It should be added that we frequently prefix an operator like (4.2) or (4.5) to sentential functions which contain – besides "x" – other free variables (this occurs in nearly all cases in which such operators are applied in geometry). The expressions thus obtained, for instance:

$$\textit{The set of all numbers } x \textit{ such that } x > y \tag{4.8}$$

do not designate, however, any definite class; they are designatory functions in the meaning established in Section 1.2, that is, they become designations of classes if we replace in them free variables (but not "x") by suitable constants. For instance, replacing "y" by "0" yields (4.6). One should now, after this reading, inspect Exercises 12 and 13 from Chapter 3 once more.

It is frequently said of a sentential function with one free variable that it expresses a certain property of objects, a property possessed by those, and only those, objects which satisfy the sentential function. For example, the sentential function:

$$x \textit{ is divisible by } 2,$$

expresses a certain property of the number x, namely, divisibility by 2, or the property of being even. The class corresponding to this function contains as its elements all objects possessing the given property, and no others. In this manner it is possible to correlate a uniquely determined class with every property of objects. And also, conversely, with every class there is correlated a property possessed exclusively by the elements of the class, namely, the property of belonging to that class. It is, accordingly, in the opinion of numerous logicians, unnecessary to distinguish at all between the two concepts of a class and of a property; in other words, a special "theory of properties" is dispensable, the theory of classes is sufficient.

As an application of these remarks we shall give a new formulation of Law 3.1, Leibniz's Law. The original one contained the term "*property*"; in the following, equivalent, formulation we employ the term "*class*" instead.

Law 4.1 (The Law of Identity)**.** *$x = y$ if, and only if, every class which contains any one of the objects x and y as an element also contains the other as an element.*

As can be seen from this formulation of Leibniz's Law, it is possible to define the concept of identity in terms of the theory of classes.

4.3 Universal and Null Classes

As we already know, to any sentential function with one free variable there corresponds the class of all objects satisfying this function. This can now be applied to the following two particular functions:

$$x = x, \quad x \neq x. \tag{4.9}$$

The first of these functions is obviously satisfied by every individual (cf. Section 3.2). The corresponding class,

$$\underset{x}{\mathrm{C}}(x = x), \tag{4.10}$$

therefore, contains as elements all individuals.

Definition 4.2. The *universal class* contains all elements, is defined by Sentence (4.10), and is denoted by the symbol "U".

The second sentential function in (4.9), on the other hand, is satisfied by no object, since, $x \neq x$ is defined by $\sim x = x$. The corresponding class is:

$$\underset{x}{\mathrm{C}}(x \neq x), \tag{4.11}$$

Definition 4.3. The *null class* (or *empty set*) contains no elements, is defined by Sentence (4.11), and is denoted by the symbol "$\emptyset$".

We may now replace the sentential function in (4.9) by equivalent functions of the form given in (4.5):

$$x \in U, \quad x \in \emptyset, \tag{4.12}$$

the first of which is satisfied by any individual, and the second by none.

It should be emphasized that U is the class of all individuals but not the class containing as elements all possible objects, thus also classes of the first order, second order, and so on. The question arises whether such a class of all possible objects exists at all, and more generally, whether we may consider "inhomogeneous" classes not belonging to a particular order and containing as elements individuals as well as classes of various orders. This question is closely related to the more intricate problems of contemporary logic, namely *Russell's Antimony* (or *Russell's Paradox*) and the *Theory of Logical Types*.[2] A discussion of this question would trespass beyond the intended limits of this book. However, to alleviate the curiosity of the reader, consider the following scenario: if we define B to be the set of all sets which are not members of themselves, then B is not a member of itself, which then implies that it is a member of itself. In logical formalism:

$$B \in B \leftrightarrow \sim B \in B.$$

4.4 Fundamental Relations Among Classes

Various relations may hold between two sets (or classes) K and L.

Definition 4.4. If every element of the set (class) K is at the same time an element of the set (class) L, then K is said to be a *subset* (*subclass*) of L. This relation among classes can be expressed by the formula:

$$K \subseteq L$$

Definition 4.5. If L is a subclass of K, then K is said to be a *superclass* of L, and is denoted by:

$$K \supseteq L$$

By saying that K is a subclass of L it is not intended to preclude the possibility of L also being a subclass of K. In other words, K and L may be subclasses of each other and thus have all the same elements; in this case it follows from a law (given below) in the theory of classes that K and L are identical.

Definition 4.6. If every element of the class K is an element of the class L, but if not every element of the class L is an element of the class K, then K is said to be a *proper subset*, or *proper subclass*, of L. This relation is denoted as follows:

$$K \subset L.$$

Definition 4.7. Two classes K and L are said to *overlap*, or to *intersect*, if they have at least one element in common and if, at the same time, each contains elements not contained in the other.

Definition 4.8. Two classes K and L are said to be *disjoint* if both classes have at least one element (i.e. both classes are non-empty), and they share no common elements.

We illustrate these set relations with some examples. The set of all integers is a proper subset of the rational numbers, while a line is a superset of any segment of the line. A circle, for instance, intersects any straight line drawn through its center, bit it is disjoint from any straight line whose distance from the center is greater than the radius. The set of all positive numbers and the set of all rational numbers overlap, but the set of positive and the set of negative numbers are mutually exclusive, and are therefore considered to have the relation of disjointness.

For more concrete examples all of the following are true statements:

$$\begin{aligned}
\{\text{apple}, \text{table}\} &\subset \{\text{apple}, \text{table}, \text{penny}\} \\
\{\text{apple}, \text{table}, \text{penny}\} &\subseteq \{\text{apple}, \text{table}, \text{penny}\} \\
\{\alpha, \beta, \gamma, \delta\} &\supset \{\alpha, \beta, \gamma\} \\
\{\alpha, \beta, \gamma, \delta\} &\supseteq \{\alpha, \beta, \gamma, \delta\} \\
\{\text{apple}, \text{table}\} \text{ overlaps with } &\{\text{apple}, \text{penny}\} \\
\{\text{apple}, \text{table}\} \text{ is disjoint from } &\{\alpha, \beta, \gamma, \delta\} \\
\{\alpha, \beta, \gamma, \delta\} &= \{\gamma, \beta, \alpha, \delta\}
\end{aligned}$$

We next give some important examples of laws concerning the relations between classes just defined.

Law 4.2 (Law of Class Reflexivity for Inclusion)**.** $\forall K \ (K \subseteq K)$

Law 4.3. $(K \subseteq L \wedge L \subseteq K) \rightarrow K = L$

Law 4.4 (Law of Class Transitivity for Inclusion)**.**

$$(K \subseteq L \wedge L \subseteq M) \rightarrow K \subseteq M$$

Law 4.5. *If K is a non-empty subclass of L, and if the classes L and M are disjoint, then the classes K and M are disjoint.*

Laws 4.4 and 4.5 and others of a similar structure form a group of statements which are called the *Laws of Categorical Syllogism*.

A characteristic property of the universal and null classes in connection with the concept of inclusion is expressed in the following law:

Law 4.6. $\forall K \ (K \subseteq U \wedge \emptyset \subseteq K)$

This statement, particularly in view of its second part referring to the empty set (null class), seems to many people somewhat paradoxical. In order to demonstrate this second part, let us consider the following implication which is the definition of subset with resect to objects in the sets:

Definition 4.9. $K \subseteq L \overset{def}{\longleftrightarrow} (x \in K \rightarrow x \in L)$

Using the above element definition of the relation subset among classes, we are now in a position to logically prove any laws and theorems which rely on this relation. For instance, we can now prove Law 4.4:

Law 4.4. $(K \subseteq L \wedge L \subseteq M) \to K \subseteq M$

Proof. Since it is a conditional sentence, we can assume the hypothesis.

(1)	$K \subseteq L \wedge L \subseteq M$	Assume hypothesis
(2)	$K \subseteq L \leftrightarrow (\, x \in K \to x \in L\,)$	Definition 4.9
(3)	$L \subseteq M \leftrightarrow (\, x \in L \to x \in M\,)$	Inst. of Definition 4.9, $K : L, L : M$
(4)	$(\, x \in K \to x \in L\,) \wedge (\, x \in L \to x \in M\,)$	Substitute (2), (3) into (1)
(5)	$[\,(\, x \in K \to x \in L\,) \wedge (\, x \in L \to x \in M\,)\,] \to (\, x \in K \to x \in M\,)$	Inst. of Law 2.5 (Hyp. Syll.) $p : x \in K, q : x \in L, r : x \in M$
(6)	$x \in K \to x \in M$	Rule of Detachment: (4), (5)
(7)	$K \subseteq M \leftrightarrow (\, x \in K \to x \in M\,)$	Instance of Definition 4.9, with $L : M$
(8)	$K \subseteq M$	Substitute (7) into (6)

□

Without performing a rigorous proof, Law 4.6 is easy to prove as well. If we substitute $\emptyset$ for K, and K for L in Definition 4.9, we arrive at:

$$\emptyset \subseteq K \leftrightarrow (\, x \in \emptyset \to x \in K\,) \tag{4.13}$$

The right side of the biconditional in equation (4.13) is always true. To see this, note that the antecedent is $x \in \emptyset$, which is false always. Thus, the entire conditional sentence $x \in \emptyset \to x \in K$ is true, which is the definition of a set being a subset of another, thus $\emptyset \subseteq K$ is true for any set K. In an analogous way, the first half of the conjunction in Law 4.6 can be demonstrated.

It is easy to see that between any two classes one of the relations considered here has to hold; the following theorem is to this effect:

Theorem 4.1. *If K and L are two arbitrary classes, then either $K = L$, K is a proper subclass of L, L is a proper subclass of K, K and L overlap, or finally, K and L are disjoint; no two of these relations can hold simultaneously.*

In order to get a clear and intuitive understanding of this law it is best to think of the classes K and L as geometrical figures and to imagine all the possible positions in which these two figures may be oriented with respect to each other, which in set theory is known as a *Venn Diagram*.

The relations which have been dealt with in this section may be called the *Fundamental Relations Among Classes*.[3]

The whole of the old traditional logic (cf. Section 2.1) can almost entirely be reduced to the theory of the fundamental relations among classes, that is, to a small fragment of the entire theory of classes. Outwardly these two disciplines differ by the fact that, in the old logic, the concept of a class does not appear explicitly. Instead of saying, for instance, that the class of horses is contained in the class of mammals, one used to say in the old logic that the property of being a mammal belongs to all horses, or simply, that every horse is a mammal. The most important laws of traditional logic are those of the categorical syllogism which correspond precisely to the laws of the theory of classes that we stated above and named after them. For example, the first of the laws of syllogism, Law 4.4, assumes the form in the old logic of:

If every M is P and every S is M, then every S is P.

This is the most famous of the laws of traditional logic, known as the *Law of Syllogism Barbara*. From the time of Aristotle, deductive reasoning was applied to arrive at conclusions based on two or more propositions. The premises were defined as major and minor, and all combinations of these conditional premises along with universal and existential quantifiers were studied to determine which conclusions can be drawn. The *Law of Syllogism Barbara*, in classical form, states "If all M are P and all S are M, then all S are P". Here, "all M are P"is the major premise, "all S are M" is the minor premise, and " every S is P" is the conclusion. As another example, the *Law Syllogism Boroco* is of the form:

If all P are M and some S are not M, then some S are not P.

In all, there are 24 valid (true) syllogisms with major and minor quantified conditional premises containing three properties S, M and P, implying a quantified conditional conclusion.

4.5 Operations on Classes

We shall now concern ourselves with certain operations, which, if performed on given classes, yield new classes. Similar to the operations of addition and multiplication of real numbers, each of the following two operations will create a new class from two given classes.

Definition 4.10. Given any two classes K and L, the *union* (or *addition*) of the two classes forms a new class M which contains as its elements those, and only those, objects which belong to at least one of the classes K and L, and is expressed symbolically as:

$$M = K \cup L.$$

Definition 4.11. Given any two classes K and L, the *intersection* or (*multiplication*) of the two classes forms a new class M which contains as its elements those, and only those, objects which belong to both of the classes K and L, and is expressed symbolically as:

$$M = K \cap L.$$

These two operations are frequently applied in geometry; sometimes it is very convenient to define, with their help, new kinds of geometrical figures. Suppose, for instance, we know already what is meant by a pair of supplementary angles; then the half-plane – that is, the straight angle – may be defined as the union of two supplementary angles (an angle here being considered as an angular region, that is, as a part of the plane, bounded by the two half-lines which are called the legs of an angle). Or, if we take an arbitrary circle and an angle whose vertex lies in the center of the circle, then the intersection of these two figures is a figure called a circular sector.

Let us add two more examples from the field of arithmetic: the union of the set of all positive numbers and of the set of all negative numbers is the set of all numbers different from 0; the intersection of the set of all even numbers and of the set of all prime numbers is the set having as its sole element the number 2, this number being the only even prime number. We next give more concrete examples of unions and intersections:

$$\begin{aligned}
\{\text{apple}, \text{table}\} \cap \{\text{apple}, \text{table}, \text{penny}\} &= \{\text{apple}, \text{table}\} \\
\{\text{apple}, \text{table}\} \cup \{\text{apple}, \text{table}, \text{penny}\} &= \{\text{apple}, \text{table}, \text{penny}\} \\
\{0, 2, 4, 6, 8, 10, \ldots\} \cap \{0, 5, 10, 15, 20, 25, \ldots\} &= \{0, 10, 20, \ldots\} \\
\{0, 2, 4, 6, 8, 10, \ldots\} \cap \{1, 3, 5, 7, 9, 11, \ldots\} &= \emptyset \\
\{-6, -4, -2, 0, 2, 4, 6, \ldots\} \cup \{-5, -3, -1, 1, 3, 5, \ldots\} &= \mathbb{Z}
\end{aligned}$$

The union and intersection of classes are governed by various laws. Some of these are completely analogous to the corresponding theorems of arithmetic concerning the addition and multiplication of numbers; as an example we mention the following two laws:

Law 4.7 (Commutative Law of Union for Classes)**.** $\forall K, L\ (K \cup L = L \cup K)$

Law 4.8 (Commutative Law of Intersection for Classes)**.**

$$\forall K, L\ (K \cap L = L \cap K)$$

We will prove the first of these laws, with the help of basic logical structures, but first we must introduce an element definition corresponding to Definition 4.10:

Definition 4.12. $(x \in K \cup L) \overset{def}{\longleftrightarrow} [(x \in K) \vee (x \in L)]$

Law 4.7. $K \cup L = L \cup K$

Proof. First we note that we have removed the universal quantifier in front of Law 4.7, since K and L are arbitrary sets which do not depend on each other, this is allowed. At each step in the following proof, we assume K and L are simply arbitrary fixed sets.

(1)	$(x \in K \cup L) \leftrightarrow [(x \in K) \vee (x \in L)]$	Definition of $K \cup L$
(2)	$[(x \in K) \vee (x \in L)] \leftrightarrow [(x \in L) \vee (x \in K)]$	Law 2.11
(3)	$(x \in L \cup K) \leftrightarrow [(x \in L) \vee (x \in K)]$	Instance of Def. 4.12 with $K : L, L : K$
(4)	$[(x \in K) \vee (x \in L)] \leftrightarrow (x \in L \cup K)$	Substitute (3) into (2)
(5)	$(x \in K \cup L) \leftrightarrow (x \in L \cup K)$	Substitute (4) into (1)

□

Note that (5) is Law 4.7 in terms of elements. Thus, we may conclude that Law 4.7 is proven. Law 4.8 can be proven similarly, and you will attempt to prove many important laws in a similar fashion in the exercises. We now continue introducing more important laws of class relations and operations.

Law 4.9 (Associative Law of Union for Classes)**.**

$$\forall K, L, M \ (K \cup (L \cup M) = (K \cup L) \cup M)$$

Law 4.10 (Associative Law of Intersection for Classes)**.**

$$\forall K, L, M \ (K \cap (L \cap M) = (K \cap L) \cap M)$$

The analogy with the corresponding arithmetical theorems becomes evident when we replace the symbols "$\cup$" and "$\cap$" by "+" and "−".

Other laws, however, deviate considerably from those of arithmetic; for example consider the *Law of Tautology*:

Law 4.11 (Law of Tautology for Class Intersection)**.** $\forall K \ (K \cap K = K)$

Law 4.12 (Law of Tautology for Class Union)**.** $\forall K, \ (K \cup K = K)$

These laws become obvious on reflecting upon the meaning of the symbols "$K \cup K$" and "$K \cap K$"; if for instance, one adds to the elements of the class K the elements of the same class, one does not really add anything, and the resulting class is again the same class K. We will, however, prove Law 4.12 in a manner similar to that of Law 4.7. Before we do this, however, we introduce the element version of two sets being equal. (We saw a glimpse of this in the proof of Law 4.8.)

Definition 4.13. $(K = L) \overset{def}{\longleftrightarrow} [x \in K \leftrightarrow x \in L]$

Law 4.12. $K \cup K = K$

Proof.

(1)	$(x \in K \cup K) \leftrightarrow [(x \in K) \vee (x \in K)]$	Inst. of Def. $K \cup L$, $L : K$
(2)	$[(x \in K) \vee (x \in K)] \leftrightarrow (x \in K)$	Inst. of Law 2.10, $p : x \in K$
(3)	$(x \in K \cup K) \leftrightarrow (x \in K)$	Substitute (2) into (1)
(4)	$(K \cup K = K) \leftrightarrow [x \in K \cup K \leftrightarrow x \in K]$	Instance of Def. 4.13 $K : K \cup K, L : K$
(5)	$K \cup K = K$	Substitute (4) into (3)

□

We want to mention one other operation, which differs from those of union and intersection inasmuch as it can be performed, not on two classes, but only on one class, and therefore, is a unary operator on classes.

Definition 4.14. Given a class K the *complement* of the class K is the class of all objects not belonging to the class K. The complement of the class K is denoted by:

$$K'$$

If K, for instance, is the set of all integers, then all fractions and irrational numbers belong to the set K'.

As examples of laws which concern the concept of complement and establish this connection with concepts considered earlier, we give the following two laws:

Law 4.13 (Law of Excluded Middle for Classes)**.** $\forall K\ (K \cup K' = U)$

Law 4.14 (Law of Contradiction for Classes)**.** $\forall K\ (K \cap K' = \emptyset)$

The relations between classes and the operations on classes with which we have just become acquainted, and also the concepts of the universal class and the null class, are treated in a special part of the theory of classes; since the laws concerning those relations and operations tend to have the character of simple formulas reminiscent of those of arithmetic, this part of the theory is known as the *Calculus of Classes*.

4.6 Cardinal Number of a Class

Among the remaining concepts which form the subject of investigation of the theory of classes there is one group which deserves particular attention and which comprises such concepts as equinumerous classes, cardinal number of a class, finite and infinite classes. They are, unfortunately, rather involved concepts which can only be superficially discussed here.

As an example of two *equinumerous* or *equivalent classes*, we may consider the set of the fingers of the right and of the left hands; these sets are equinumerous, because it is possible pair off the fingers of both hands in such a manner that (i) every finger occurs in just one pair, and (ii) every pair contains just one finger from the left hand and just one finger from the right hand. In a similar sense, the following three sets are equinumerous: the set of all vertices, the set of all sides, and the set of all angles of a polygon. Later in Section 5.7, we shall be able to give an exact general definition of this concept of equinumerous classes.

Now let us consider an arbitrary class K; there exists, no doubt, a property belonging to all classes equinumerous to K and to no other classes (namely, the property of being equinumerous with K); this property is defined as follows:

Definition 4.15. The *cardinal number* of a class K is the class of all classes equinumerous with K.

It follows from this that two classes K and L have the same cardinal number if, and only if, they are equinumerous.

With regard to the number of their elements, classes are classified into finite and infinite ones. Among the former, we distinguish between classes consisting of exactly one element, of two, of three elements, and so on. These terms are most easily defined on the basis of arithmetic.

Definition 4.16. Let n be an arbitrary natural number (that is, a positive integer); then *the class K consists of n elements* if K is equinumerous with the class of all natural numbers less than or equal to n.

In particular, a class consists of 2 elements, if it is equinumerous with the class of all natural numbers less than 2, i.e. to the class consisting of the two numbers 1 and 2. Similarly, a class consists of 3 elements if it is equinumerous with the class containing 1, 2, and 3 as its elements. This leads us to the following definition:

Definition 4.17. A class K is *finite* if there exists a natural number n such that the class K consists of n elements.

It should be noted that there is one finite case for which our definition does not hold. The empty set (null class) has 0 elements, which we consider to be a special case. There is only one class with 0 elements, and it is the null class.

Definition 4.18. If a class is not finite, then by definition, it is *infinite*.

It has, however, been recognized that there is still another possible procedure. All the terms which have just been considered can be defined in purely logical terms without resorting at all to any expressions belonging to the field of arithmetic. We may, for instance, say that the class K consists exactly one element, if this class satisfies the following two conditions: (i) there is an x such that $x \in K$; (ii) for any y and z, if $y \in K$ and $z \in K$, then $y = z$. In quantified form, this can be expressed as:

$$[\,\exists\, x\ (x \in K)\,] \wedge [\,\forall\, y, z\ (\,(y \in K \wedge z \in K) \rightarrow y = z\,)\,]\,.$$

These two conditions may also be replaced with the quantified Sentence 3.8 from Section 3.5. Analogously, we can define the phrases: "The class K consists of two elements", "the class K consists of three elements", and so on. The problem becomes more difficult when we turn to the question of defining the terms "finite class" and "infinite class"; but also in these cases the efforts of solving the problem positively have been successful (cf. Section 5.7), and thereby the concepts under consideration have been included within the range of logic.

This circumstance has a most interesting consequence of far reaching importance; for it turns out that the notion of number itself and likewise all other arithmetical concepts are definable within the field of logic. It is, indeed, easy to establish the meaning of symbols designating individual natural numbers, such as "0", "1", "2", and so on. The number 1, for instance, can be defined as the number of elements of a class which consists of exactly one element. (A definition of this kind seems to be incorrect and contains apparently a vicious circle, since the word "one", which is about to be defined, occurs in the definiens; but actually no error is committed because the phrase "the class consists of exactly one element" is considered as a whole and its meaning has been defined previously.) Nor is it hard to define the general concept of a natural number: a natural number is the cardinal number of a finite class. We are, further, in a position to define all operations on natural numbers, and to extend the concept of numbers by the introduction of fractions, negative and irrational numbers, without, at any place, having to go beyond the limits of logic. Furthermore, it is possible to prove all the theorems of arithmetic on the basis of laws of logic alone (with the qualification that the system of logical laws must first be enriched by the inclusion of a statement which is intuitively less evident than the others, namely, the so-called *Axiom of Infinity*, which states that there are infinitely many different objects). This entire construction is very abstract, it cannot be easily be popularized and does not fit into the framework of an elementary presentation of arithmetic; in this book, we also do not attempt to adopt ourselves to this conception and treat numbers as individuals and not as properties of classes of classes. But the mere fact that it has been possible to develop the whole of arithmetic, including the disciplines erected upon it, algebra, analysis, and so on, as a part of pure logic, constitutes one of the grandest achievements of logical investigations of the 20th century.[4]

Exercises

1. Let K be the set of all numbers less than $\frac{3}{4}$; which of the following sentences are true?
   (a) $0 \in K$ (b) $1 \in K$ (c) $\frac{2}{3} \in K$
   (d) $\frac{3}{4} \in K$ (e) $\frac{4}{5} \in K$ (f) $-\frac{2}{3} \in K$

2. Consider the following four sets:
   (a) the set of all positive numbers,
   (b) the set of all numbers less than 3,
   (c) the set of all numbers x such that $x + 5 < 8$,
   (d) the set of all numbers x satisfying the sentential function "$x < 2x$".

Which of these sets are identical, and which are distinct?

3. What name in geometry is given to the set of all points whose distance from a given point does not exceed the length of a given line segment?

4. What name in geometry is given to the set of all points whose distance from a given point is equal to the length of a given line segment?

5. Let K and L be two concentric disks, the radius of the first being smaller than that of the second. Which of the relations discussed in Section 4.4 hold between these disks?

6. Repeat problem 5, but this time using only the circumference of the disk (i.e. consider two concentric circles).

7. Let x and y be two arbitrary numbers, with $x < y$. It is well known that the set of numbers which are not smaller than x and not larger than y is called the *closed interval* with endpoints x and y; it is denoted by the symbol "$[x, y]$". State which of the formulas below are correct:
   (a) $[3,5] \subseteq [3,6]$ (b) $[3,5] \subseteq [3,5]$ (c) $[3,5] \subset [3,5]$
   (d) $[4,7] \subseteq [5,10]$ (e) $[-2,4] \supseteq [-3,5]$ (f) $[-2,4] \supseteq [1,2]$
   (g) $[-2,4] \supset [-2,4]$ (h) $[-7,1] \supseteq [-5,-2]$ (i) $0 \subseteq [-2,4]$

8. Translate the following formulas into terms of ordinary language:
   (a) $(x = y) \leftrightarrow \forall K\ [\,(x \in K) \leftrightarrow (y \in K)\,]$
   (b) $(K = L) \leftrightarrow \forall x\ [\,(x \in K) \leftrightarrow (x \in L)\,]$

9. What alterations on both sides of the equivalence from Exercise 8 (b) would be required in order to arrive at the definitions of the symbols "$\subseteq$" and "$\supseteq$"?

10. What alterations on both sides of the equivalence from Exercise 8 (b) and your answers to Exercise 9 would be required in order to arrive at the definitions of the symbols "$\subset$" and "$\supset$"?

11. Determine which of the following formulas are true:

(a) $[2,4] \cup [3,5] = [2,5]$ (b) $[-1,2] \cup [0,3] = [0,2]$

(c) $[-2,8] \cap [3,7] = [-2,8]$ (d) $[2,4] \cup [5,6] = [2,6]$

(e) $[-2,8] \cap [3,7] \subseteq [-2,7]$ (f) $[2,7] \cap [2,5] \supseteq [2,3]$

12. In the formulas which are false in the previous exercise, correct the expressions on the right side of the class symbol relation symbol.

13. We can express the definition of the union of two classes K and L in terms of their elements as follows (cf. Definition 4.12):

$$(x \in K \cup L) \overset{def}{\longleftrightarrow} [(x \in K) \vee (x \in L)]$$

Formulate, analogously, the definitions of intersection and complement, as well as the definitions of the universal class and the empty set.

14. Is there a polygon, in which the set of all sides is equinumerous with the set of all diagonals?

15. Lay down definitions of the following expressions, using terms from the field of logic exclusively:

(a) *The class K has at least two elements.*
(b) *The class K has at most three elements.*
(c) *The class K consists of exactly three elements.*

Notes

[1]The beginnings of the theory of classes – or, to be more exact, of that part of this theory which we shall denote as the calculus of classes below – are already found in G. Boole (cf. endnote 1 of Chapter 1). The actual creator of the general theory of sets as an independent mathematical discipline was the great German mathematician G. Cantor (1845-1918); we are indebted to him, in particular, for the analysis of such concepts as equality in power, cardinal number, infinity, and order, which will be discussed in the course of the present and the next chapters. – Cantor's set theory is one of those mathematical disciplines which are in a state of especially intensive development. Its ideas and lines of thought have penetrated into almost all branches of mathematics and have exerted everywhere a most stimulating and fertilizing influence.

[2]The concept of logical types introduced by Russell is akin to that of the order of a class, and can even be conceived as a generalization of the latter, – a generalization which refers not only to classes but also to other things, for instance, to relations, which will be considered in the next chapter. The theory of logical types was systematically developed in *Principia Mathematica* (cf. endnote 1 of Chapter 1).

[3]These relations were first investigated in an exhaustive manner by the French mathematician J.D. Gergonne (1771-1859).

[4]The fundamental ideas in this field are due to Frege (cf. footnote 2 of Chapter 1); he developed them for the first time in his interesting book: *Die Grundlagen der Arithmetik* (Breslau 1884). Frege's ideas found their systematic and exhaustive realization in Whitehead and Russel's *Principia Mathematica* (cf. footnote 1 of Chapter 1).

Chapter 5

On The Theory of Relations

5.1 Relations and Sentential Functions with Two Free Variables

In the previous chapters we were introduced to a few relations between objects. As examples of relations between two objects we may take, for instance, identity (equality) and diversity (inequality). We sometimes read the formula

$$x = y$$

as follows:

x has the relation of identity to y,

or also:

the relation of identity holds between x and y,

and we say that the symbol "=" designates the relation of identity. In an analogous way, the formula:

$$x \neq y$$

is sometimes read:

x has the relation of diversity to y,

or:

the relation of diversity holds between x and y,

and one says that the symbol "≠" designates the relation of diversity. We have further encountered certain relations holding between classes, namely, the relations of inclusion, of overlapping, of disjointness, and so on. We will now

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discuss several concepts belonging to the general *Theory of Relations*, which constitutes a special and very important part of logic, and in which relations of an entirely arbitrary character are considered and general laws concerning them are established.[1]

To facilitate our considerations, we introduce special variables "R", "S", ... which serve to denote relations. In place of phrases as:

the object x has the relation R to the thing y,

and

the object x does not have the relation R to the thing y,

we shall employ symbolic abbreviations:

$$xRy$$

and (to use the negation sign of sentential calculus, cf. Section 2.8)

$$\sim xRy,$$

respectively.

Definition 5.1. Any object x having the relation R to some object y we call a *predecessor with the respect to the relation R.*

Definition 5.2. Any object y for which there is a thing x such that xRy is called a *successor with respect to the relation R.*

Definition 5.3. The class of all predecessors "x" with respect to the relation R is known as the *domain* of the relation R.

Definition 5.4. The class of all successors "y" with respect to the relation R is known as the *counter-domain* of the relation R.

For example, any individual is both a predecessor and a successor with respect to the relation of identity, so that the domain and counter-domain are both the universal class U.

In the theory of relations – just as in the theory of classes – we may distinguish relations of different orders. The *relations of the first order* are those which hold between individuals; *the relations of the second order* are those which hold between classes, or relations of the first order; and so on. The situation is here all the more complicated as we must often consider "mixed" relations whose predecessors are, say, individuals, and its successors classes, or whose predecessors are, for instance, classes of the first order and its successors classes of the second order. The most important example of a relation of this kind is the relation which holds between an element and a class to which it belongs; as we recall from Section 4.1, this relation is denoted by the symbol

"$\in$". As in the case of classes, our considerations concerning relations will refer primarily to those of the first order, although the concepts discussed here can and, in a few cases, will be applied to relations of higher orders.

We assume that, to every sentential function with two free variables "x" and "y", there corresponds a relation holding between the objects x and y if, and only if, they satisfy the given sentential function; in this connection it is said of a sentential function with the free variables "x" and "y" that it expresses a relation between the objects x and y. Thus, for instance, the sentential function:

$$x + y = 0$$

expresses the relation of having the opposite sign or, briefly, of being opposite; the numbers x and y have the relation of being opposite if, and only if, $x+y = 0$. If we denote this relation by the symbol "O", then the formulas:

$$xOy, \quad x + y = 0$$

are equivalent. Similarly, any sentential function containing the symbols "x" and "y" as the only free variables may be transformed into an equivalent formula of the form xRy, where in place of "R", we have a constant which designates some relation. The formula xRy may, therefore, be considered as the general form of a sentential function with two free variables, just as the formula $x \in K$ could be looked upon as the general form of a sentential function with one free variable (cf. Section 4.2).

5.2 Calculus of Relations

The theory of relations is one of the farthest developed branches of mathematical logic. One part of it, the *Calculus of Relations*, is akin to the calculus of classes, its principal object being the establishment of formal laws governing the operations by means of which other relations are constructed from given ones.

In the calculus of relations we consider, in the first place, a group of concepts which are exact analogues of those of the calculus of classes; they are usually denoted by the same symbols and governed by quite similar laws. In order to avoid ambiguity, we might, of course, employ a different set of symbols in the calculus of relations, taking, for instance, the symbols of the calculus of classes and placing a dot over each.

We have thus in the calculus of relations two special relations, the *universal relation* U and the *null relation* $\emptyset$, the first of which holds between any two individuals, and the second between none.

We have, further, various relations between relations, consider, for instance, the *relation of inclusion*:

Definition 5.5. The relation R is *included* in the relation S, symbolically:

$$R \subseteq S,$$

if, whenever R holds between two objects, S holds between them as well. In other words,

$$\forall\, x, y\ (xRy \to xSy).$$

We know, for instance, from arithmetic that:

$$x < y \to x \neq y,$$

hence the relations of $<$ and $\neq$ satisfy the relation $< \subseteq \neq$. In words, we state the relation of being smaller (lesser) is included in the relation of diversity (or not being equal).

Definition 5.6. If at the same time, $R \subseteq S$ and $S \subseteq R$, that is to say, if the relations R and S hold between the same objects, then we say that the relations are *identical*, and is denoted:

$$R = S,$$

and in terms of elements:

$$\forall\, x, y\ (xRy \leftrightarrow xSy).$$

We also have operations on relations, just as with classes.

Definition 5.7. The *union* (or *sum*) of two relations R and S, is a new relation which holds between two objects if, and only if, at least one of the relations R and S hold between them. The relation is denoted symbolically by:

$$R \cup S,$$

and in terms of elements by:

$$xR \cup Sy \xleftrightarrow{def} xRy \vee xSy$$

Definition 5.8. The *intersection* (or *product*) of two relations R and S, is a new relation which holds between two objects if, and only if, both of the relations R and S hold between them. The relation is denoted symbolically by:

$$R \cap S,$$

and in terms of elements by:

$$xR \cap Sy \xleftrightarrow{def} xRy \wedge xSy$$

Both the union and intersection of relations are binary operations. Thus, for example, if R is the relation of fatherhood (that is, a relation holding between two persons x and y if, and only if, x is the father of y), and S the relation of motherhood, then $R \cup S$ is the relation of parenthood, while $R \cap S$ is, in this case, the null relation.

Definition 5.9. The unary relation operation of *negation* (or *complement*) is denoted as:

$$R',$$

and is defined to be the relation which holds between two objects if, and only if, the relation R does not hold between them; in other words,

$$xR'y \overset{def}{\longleftrightarrow} \sim xRy$$

It should be noted that, if a relation is designated by a constant, then its complement is frequently denoted by the symbol obtained from that constant by crossing it by a vertical or oblique bar. The negation of the relation $<$, for example, is usually denoted by "$\not<$", and not by "$<'$".

In the calculus of relations there occur also entirely new concepts, without analogues in the calculus of classes.

We have here, first, two special relations, *identity* and *diversity* between individuals (which are, incidentally, familiar to us from earlier considerations). In the calculus of relations, they are denoted by special symbols, e.g., "I" and "D", and not by the symbols "$=$" and "$\neq$" used in other parts of logic. We write thus:

$$xIy \text{ and } xDy$$

instead of:

$$x = y \text{ and } x \neq y.$$

The symbols "$=$" and "$\neq$" are used in the calculus of relations only to denote the identity and diversity between relations.

We have here, further, a very interesting and important new operation, denoted by the symbol "/", with the help of which we form, from two relations R and S, a new relation R/S.

Definition 5.10. The *relative product* of R and S, denoted by:

$$R/S,$$

holds between two objects x and y if, and only if, there exists an object z such that we have at the same time

$$xRz \text{ and } zSy.$$

We define this precisely as:

$$xR/Sy \overset{def}{\longleftrightarrow} \exists\, z\ (xRz \wedge zSy)$$

Thus, for instance, if R is the relation of being husband and S the relation of being daughter, then R/S holds between two persons x and y if there is a person z such that x is the husband of z and z is the daughter of y; the relation R/S, therefore, coincides with the relation of being son-in-law. Another operation of a similar character, whose result is called the *relative sum*, does not play any role in the future discussion of this text and will not be defined here.

Finally, we introduce another unary operation on classes (the first being the complement relation).

Definition 5.11. Given a relation R, the *converse* relation to R is denoted as

$$\breve{R},$$

and holds between x and y if, and only if, R holds between y and x. Formally we have

$$x\breve{R}y \overset{def}{\longleftrightarrow} yRx.$$

If a relation is denoted by a constant, then for denoting its converse we often employ the same symbol printed in the opposite direction. The converse of the relation $<$, for instance, is the relation $>$, since, for any x and y, the formulas:

$$x < y$$

and

$$y > x$$

are equivalent.

In view of the rather specialized character of the calculus of relations, we shall here not go any further into the details of it.

5.3 Some Properties of Relations

We now turn to that part of the theory of relations whose task it is to single out and investigate special kinds of relations with which one meets frequently in other sciences and, in particular, in mathematics.

Definition 5.12. A relation R is *reflexive in the class* K if every element x of the class K has the relation R to itself:

$$R \text{ is } \textit{reflexive} \overset{def}{\longleftrightarrow} \forall x \in K\ (xRx).$$

Definition 5.13. A relation R is *irreflexive in the class* K if no element x of the class K has the relation R to itself:

$$R \text{ is } \textit{irreflexive} \overset{def}{\longleftrightarrow} \forall x \in K\ (\sim xRx).$$

It should be noted that with the help of the complement relation, we can also define irreflexive as:

$$R \text{ is } \textit{irreflexive} \overset{def}{\longleftrightarrow} \forall\, x \in K\ (xR'x).$$

Definition 5.14. A relation R is *symmetrical in the class* K if for any two elements x and y, the formula xRy implies the formula yRx:

$$R \text{ is } \textit{symmetrical} \overset{def}{\longleftrightarrow} \forall\, x, y \in K\ (xRy \to yRx).$$

Since we have the converse operation on a relation R, symmetrical can be also defined as:

$$R \text{ is } \textit{symmetrical} \overset{def}{\longleftrightarrow} \forall\, x, y \in K\ (xRy \to x\breve{R}y).$$

Definition 5.15. A relation R is *asymmetrical in the class* K if for any two elements x and y, the formula xRy implies that $\sim yRx$:

$$R \text{ is } \textit{asymmetrical} \overset{def}{\longleftrightarrow} \forall\, x, y \in K\ (xRy \to \sim yRx).$$

Definition 5.16. A relation R is *transitive in the class* K if for any three elements x, y, and z, the formulas xRy and yRz implies that xRz:

$$R \text{ is } \textit{transitive} \overset{def}{\longleftrightarrow} \forall\, x, y, z \in K\ (xRy \wedge yRz \to xRz).$$

And lastly:

Definition 5.17. A relation R is *connected in the class* K if for any two distinct elements x and y, then at least one of the formulas xRy and yRx holds.

$$R \text{ is } \textit{connected} \overset{def}{\longleftrightarrow} \forall\, x, y \in K\ (\sim (x = y) \to (xRy \vee yRx)).$$

Another way to describe a relation R being connected is that the relation R always holds between two arbitrary elements of K in at least one direction assuming they are distinct.

In case K is the universal class U (or, at any rate, the universe of discourse of the science in which we happen to be interested, cf. Section 4.3) we usually speak, more briefly, not of relations reflexive, symmetrical, and so on, in the class K, but simply of reflexive relations, symmetrical relations, and so on.

It should also be remarked that presupposing a relation R has one of the aforementioned properties sometimes implies that the relation R has one of the other defined relations. For instance, consider the following theorem:

Theorem. *R is asymmetrical $\to$ R irreflexive.*

Proof. In proving this theorem, pay close attention to step (7) where the universal quantifier $\forall x, y$ is reduced to a universal quantifier $\forall x$ be setting $y = x$ in a conditional setting. Since, by assumption, $\sim xRy$ holds true for all x and y, it must in particular, hold for the case $y = x$. Note that this is not a biconditional sentence, as the implication only works one way. This can be expressed in terms of the quantified statement:

$$\forall x, y\ P(x, y) \rightarrow \forall x\ P(x).$$

Of course, similar reductions can be made for universally quantified statements of three or more variables.

(1)	R asymmetrical	Assume hypothesis
(2)	R asymmetrical $\leftrightarrow \forall x, y\ (xRy \rightarrow \sim yRx)$	Definition of asymmetrical
(3)	$\forall x, y\ (xRy \rightarrow \sim yRx) \leftrightarrow \forall x, y\ (\sim xRy \vee \sim yRx)$	Instance of $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ $p : xRy,\ q : yRx$
(4)	$\forall x, y\ (\sim xRy \vee \sim yRx) \rightarrow \forall x\ (\sim xRx \vee \sim xRx)$	$\forall x, y\ P(x, y) \rightarrow \forall x\ P(x)$, $P(x, y) : \sim xRy \vee \sim yRx$
(5)	$\forall x, y\ (xRy \rightarrow \sim yRx)$	Rule of Substitution: (2) into (1)
(6)	$\forall x\ (\sim xRx \vee \sim xRx)$	Rule of Detachment: (4) and (5)
(7)	$\forall x\ (\sim xRx \vee \sim xRx) \leftrightarrow \forall x\ (\sim xRx)$	Law of Or Tautology: $p : \sim xRx$
(8)	$\forall x\ (\sim xRx)$	Rule of Substitution: (7) into (6)
(9)	R irreflexive $\leftrightarrow \forall x\ (\sim xRx)$	Definition of irreflexive
(10)	R irreflexive	Rule of Substitution: (9) into (8)

□

5.4 Relations which are Reflexive, Symmetric, and Transitive

All of the properties of relations defined in Section 5.3 frequently occur in groups. Very common, for instance, are those relations which are reflexive, symmetrical, and transitive. A typical example of this type is the relation of identity; Law 3.2 of Section 3.2 expresses that this relation is reflexive (and in fact even is called the Law of Reflexive Identity), similarly Laws 3.3 and 3.4 state that the relation of identity is symmetric and transitive, just as their titles imply. Numerous other examples of relations of this kind may be found within the field of geometry. Congruence, for instance, is a reflexive relation in the set of all line segments (or of arbitrary geometrical configurations), since every segment is congruent to itself; it is symmetrical, since, if a segment is congruent to another segment, the other is congruent to the first; and finally, it is transitive, since if the segment A is congruent to the segment B, and B to C, then the segment A is also congruent to the segment C. The same three properties belong to the relations of similarity among polygons or of parallelism among straight lines (assuming any line to be parallel to itself), or, if we consider domains outside that of geometry, to the relations of being equally old among people, or of synonymity among words.

Every relation which is at the same time reflexive, symmetrical, and transitive is thought of as some kind of equality. Instead of saying, therefore, that such a relation holds between two objects, one can, in this sense, also say that these objects are equal in such and such respect, or, in a more precise mode of speech, that certain properties of these objects are identical. Thus, instead of stating that two segments are congruent, or two people are equally old, or two words are synonymous, it may just as well be stated that the segments are equal in respect to their length, that the people have the same age, or that the meanings of the words are identical.

By way of an example we will give an indication of how it is possible to establish a logical basis for such a mode of expression. For this purpose let us consider the relation of similarity among polygons. We will denote the set of all polygons similar to the given polygon P (or, to use a slightly more current terminology, the common property which belongs to all polygons similar to P and to no others) as the shape of the polygon P. Thus shapes are certain set of polygons (or properties of polygons; cf. remarks at the end of Section 4.2). Making use of the fact mentioned above that the relation of similarity is reflexive, symmetrical, and transitive, we can now easily show that every polygon belongs to one and only one such set, that two similar polygons belong always to the same set, and that two polygons which are not similar belong to different sets. From this reasoniong, it follows at once that the two statements:

The polygons P and Q are similar.

and

The polygons P and Q have identical shapes.

are equivalent.

The reader will notice immediately that, in the course of the preceding considerations, we have once before applied an analogous procedure, namely in Section 4.6, in making the transition from the expression:

The classes K and L are equinumerous.

to the equivalent one:

The classes K and L have the same cardinal number.

It can be shown with little difficulty that the same procedure is applicable to any reflexive, symmetrical, and transitive relation. There is even a logical law, called the *Principle of Abstraction*, that supplies a general theoretical foundation for the procedure which we have been considering, but we shall here forego the exact formulation of this principle.

There is, so far, no universally accepted term denoting the totality of relations which are at the same time reflexive symmetric, and transitive. Sometimes they have generally been called *equalities* or *equivalencies*. But the term "equality" is also sometimes reserved for particular relations of the category under consideration, and two objects are then called equal if such a relation holds between them. For instance, in geometry, as has been pointed out in Section 3.4, congruent segments are often referred to as equal segments. We will emphasize here once more that it is preferable to avoid such expressions altogether; their use merely leads to ambiguities, and it violates the convention in accordance with which we consider the terms "equality" and "identity" as synonymous.

5.5 Ordering Relations

Definition 5.18. If a relation on a set K is at the same time asymmetrical, transitive and connected, we say that the relation *establishes an order in the class K*, or that *the class K is ordered by the relation R.*

Consider, for example, the relation "$<$" on any set of real numbers. First, it is asymmetrical for if x and y are two numbers such that $x < y$, it is the case that $y \not< x$. Second, if x, y, and z are three numbers, then $x < y$ and $y < z$ implies that $x < z$. Lastly, any two distinct numbers x and y must satisfy either: $x < y$ or $y < x$. All three of these properties can be visualized by placing the numbers x, y, and z on the number line in various orientations. We can therefore conclude that any set of real numbers is ordered by the relation $<$. Likewise, the relation $>$ represents another ordering for any set of real numbers.

Let us next consider the relation of being older. One can easily verify that this relation is irreflexive, asymmetrical, and transitive in any given set of people. However, it is not necessarily connected; for it can happen, perchance, that the set contains two people having exactly the same age, that is to say, who were born at the same moment, so that the relation of being older does not hold between them in either direction. If, on the other hand, we consider a set of people in which no two are of exactly the same age, the relation of being older establishes an order in that set. This example highlights the fact that the class upon which the relation is applied can influence the properties of the relation.

Many instances of relations are known that belong to neither of the two categories discussed in the present section and in the preceding one. Let us consider a few examples.

The relation of diversity is irreflexive in any set of objects, since no object is different from itself; it is symmetrical, for if $x \neq y$, then $y \neq x$. The relation fails to be transitive, since if we have $x \neq y$ and $y \neq z$, it does not imply that $x \neq z$ (to see this, simply let $x = z$ and make sure $x \neq y$). Connected is a property of the relation $\neq$, since if we pick to distinct values x and y, we have both $x \neq y$ and $y \neq x$ (due to symmetrical property and the definition of $\neq$).

The relation of inclusion $\subseteq$ between classes, by the Law 4.2, The Law of Class Reflexivity Identity, and Law 4.4, The Law of Class Transitivity for Inclusion, is reflexive and transitive. However, from Law 4.3, which defines when two sets are identical, we may conclude that the relation of inclusion is neither symmetrical nor asymmetrical (unless the aforementioned law is vacuously true only, which is not the case).

5.6 One-many Relations or Functions

We will now deal in some detail with another particularly important category of relations.

Definition 5.19. A relation R is called a *functional relation* or simply a *function* if, to every object x there corresponds at most one object y such that xRy. In other words:

$$R \text{ is a } \textit{function} \overset{def}{\longleftrightarrow} \forall x, y, z \in K \ ((xRy \wedge xRz) \rightarrow (y = z)).$$

It should be remarked that the definition of a relation being a function corresponds to the graph of a function from algebra passing the vertical line test.

Definition 5.20. All values of x for which there exists a y such that xRy constitutes the *domain*, or the *argument values*, of the function.

Definition 5.21. The set of all values y for which there exists an x such that xRy is called the *range of* R, or the *function values of* R.

If we let R be an arbitrary function, x any one of its argument values; the unique value y of the function corresponding to the value x of the domain is denoted by the symbol "$R(x)$"; consequently we replace the formula xRy by $y = R(x)$.

It has become the custom, especially in mathematics, to use, not the variables "R", "S", ..., but other letters such as "f", "g", ... to denote the functional relations, so that we find formulas like

$$y = f(x), \quad y = g(x), \ldots;$$

the formula $y = f(x)$, for instance, is read as follows:

The function f assigns the value y to the argument value x.

or:

y is the function f value corresponding to the argument value x.

In many elementary textbooks of algebra a definition of the concept of a function is to be found that is quite different from the definition adopted here. The functional relation is there characterized as a relation between two "variable" quantities or numbers: the "independent variable" and the "dependent variable", which depend upon each other in so far as a change of the first effects a change of the second.

The simplest example of a functional relation is represented by the ordinary relation of identity. As an example of a function from everyday life let us take the relation expressed by the sentential function:

$$y \text{ is the biological father of } x. \tag{5.1}$$

Clearly the relation is is a function, as for a given person x, there exists only one biological father y. In relation form, we would express this as xFy, and state this by saying: "the Father of x is y".

The concept of a function plays a most important role in the mathematical sciences. There are whole branches of higher mathematics devoted exclusively to the study of certain kinds of functional relations. But also in elementary mathematics, especially in algebra and trigonometry, we find an abundance of functional relations. Examples are the relations expressed by such formulas as:

$$\begin{aligned} y &= x + 5, \\ y &= x^2, \\ y &= \log_{10}(x), \\ y &= \sin(y), \end{aligned}$$

and many others. Let us consider the second of these formulas more closely. To every number x, there corresponds only one number y such that $y = x^2$, so that the formula really does represent a functional relation. The domain of this function is all real numbers, however the range consists only of nonnegative real numbers. If we denote this function by the symbol "f", the formula $y = x^2$ assumes the form $y = f(x)$. Evidently "x" and "y" may here be replaced by symbols designating definite numbers. For instance, $4 = (-2)^2$, so it may be asserted that $4 = f(-2)$.

On the other hand, and again in elementary mathematics already, we encountered numerous relations which are not functions. For example, the relation of being smaller is certainly not a function, since, to every number x, there are infinitely many numbers y such that $y < x$. Nor is the relation between the numbers x and y expressed by the formula:

$$x^2 + y^2 = 25$$

a functional relation, since, to one number x, there may correspond two different numbers y for which the formula is valid; corresponding to the number 4, for instance, we have both the numbers 3 and -3. It may be noted that relations between numbers which, like the one just considered, are expressed by equations that correlate with one number x two or more numbers y are sometimes called in mathematics *two-* or *many-valued* functions. It seems, however, inexpedient, at least on an elementary level, to denote such relations as functions, for this only tends to blot out the essential difference between the notion of a function and the more general one of a relation.

Functions are of particular significance as far as the application of mathematics to the empirical sciences is concerned. Whenever we inquire into the dependence between two kinds of quantities occurring in the external world, we strive to give this dependence the form of a mathematical formula, which would permit us to determine exactly the quantity of the one kind by the corresponding quantity of the other; such a formula always represents some functional relation between the quantities of two kinds. As an example let us mention the well-known formula from physics:

$$s = 16.1t^2,$$

expressing the dependence of the distance s, covered by a freely falling body, upon the time t of its fall if air resistance is neglected. Here the distance is measured in feet, and the time in seconds.

In conclusion of our remarks on functional relations we want to emphasize that the concept of a function which we are considering now differs essentially from the concepts of a sentential and of a designatory function known from Section 1.2. Strictly speaking, the terms "sentential function" and "designatory function" do not belong to the domain of logic or mathematics; they denote certain categories of expressions which serve to compose logical and mathematical statements, but they do not denote objects treated of in those statements (cf. Section 2.4). The term "function" in its new sense, on the other hand, is an expression of a purely logical character; it designates a certain type of objects dealt with in logic and mathematics. There is, no doubt, a connection between these concepts, which may be described roughly as follows. If the variable "y" is joined by the symbol "=" to a designatory function containing "x" as the only variable, e.g. to "$x^2 + 2x + 3$", then the resulting formula (which is a sentential function):

$$y = x^2 + 2x + 3$$

expresses a functional relation; or, in other words, the relation holding between those and only those numbers x and y which satisfy this formula is a function in the new sense. This is one of the reasons why these concepts are so often confused.

5.7 Bijective Functions

Definition 5.22. A function is a *bijection* if, to all elements x of the domain there is only one corresponding element y of the range, and conversely.

If f is a bijection, K an arbitrary class which is the domain of f, and L the class corresponding to the range of f, we say that the function f *maps the class K on the class L in a one-to-one manner*, or that it *establishes a one-to-one correspondence between the elements of K and L.*

Let us consider a few examples. Suppose we have a half-line issuing from the point O with a segment marked off indicating the unit length. Further let X be any point on the half-line. Then the segment OX can be measured, that is to say, one can correlate with it a certain non-negative number y called the length of the segment. Since this number depends exclusively on the position of the point X, we may denote it by the symbol "$f(X)$"; we consequently have $y = f(X)$. But conversely, to every non-negative number y, we may also construct a uniquely determined segment OX on the half-line under consideration, whose length equals y; in other words, to every y, there corresponds exactly one point X such that $y = f(X)$. The function f is, therefore, a bijection; it establishes a one-to-one correspondence between the points of the half-line and the non-negative numbers (and it would be equally simple to set up a one-to-one correspondence between the points of the entire line and all real numbers).

Another example is supplied by the relation expressed by the formula $y = x$. This is a bijection since, to every number y, there is only one number x satisfying the given formulas; it can be seen at once that this function maps, for instance, the set of all positive numbers on the set of all negative numbers in a one-to-one manner.

As a last example, we consider $y = 2x$, under the assumption that the domain is the set of natural numbers, denoted by the symbol "$\mathbb{N}$". Again we have a bijection; this function correlates with every natural number x an even number $2x$; and vice versa, i.e. to every even natural number y there corresponds just one number x such that $2x = y$, namely the number $x = \frac{1}{2}y$. The function thus establishes a one-to-one correspondence between arbitrary natural numbers and even natural numbers.

We are now in a position to lay down an exact definition of a term which, earlier on, we had only been able to characterize intuitively rather than with precision. It is the concept of equinumerous classes (see Section 4.6).

Definition 5.23. Two classes K and L are *equinumerous* if there exists a function which establishes a one-to-one correspondence between the elements of the two classes.

On the basis of this definition it follows, in connection with the examples considered above, that the set of all points of an arbitrary half-line is equinumerous with the set of all non-negative numbers; and likewise, that the set of positive numbers and the set of negative numbers are equinumerous, and that the same holds for the set of all natural numbers and the set of all even natural numbers. The last example is particularly instructive; for it shows that a class may be equinumerous with a subclass of itself. To many readers, this fact may seem most paradoxical at a first glance, because usually only finite classes are compared with respect to the number of their elements, and a finite class has, indeed, a greater cardinal number than any of its parts. The paradox disappears on calling to mind that the set of natural numbers is infinite and that we are, by no means, justified to ascribe properties to infinite classes that we have observed exclusively in connection with finite classes. It is noteworthy that the property of the set of natural numbers of being equinumerous with one of its parts is shared by all infinite classes. This property is, therefore, characteristic of infinite classes, and it permits us to distinguish them from finite classes; a finite class can simply be defined as a class which is not equinumerous with any one of its proper subclasses. (However, this definition entails a certain logical difficulty, a discussion of which we will not enter into here.)[2]

5.8 Many-Termed Relations

We have, so far, considered exclusively, binary relations. One frequently meets ternary relations, and in general, many-term relations within various sciences. In geometry, for instance, the relation of betweenness constitutes a typical example of a three-termed relation; it holds between three points of a line, and is expressed symbolically by the formula:

$$A/B/C \tag{5.2}$$

which is read:

The point B lies between the points A and C.

Arithmetic, too, supplies numerous examples of three-termed relations; it may suffice to mention the relation between three numbers x, y, and z, consisting in the fact that the first number is the sum of the other two:

$$x = y + z,$$

as well as similar relations, such as are expressed by the following formulas:

$$x = y - z,$$
$$x = y \cdot z,$$
$$x = y/z.$$

As an example of a four-termed relation let us point to the relation holding between the four points A, B, C, and D if, and only if, the distance of the first two equals the distance of the last two, in other words, if the segments AB and CD are congruent.

Of particular importance among the totality of the many-termed relations are the many-termed functional relations, which correspond to the two-termed functional relations. For reasons of simplicity we shall restrict ourselves to a discussion of the termed relations of this type.

Definition 5.24. R is called a *three-term functional relation* if, to any two things x and y, there corresponds at most one thing z having this relation to x and y, and is denoted by:

$$z = R(x, y).$$

In order to differentiate between two-termed and three-termed functional relations, we speak, in the first case, of *functions of one variable*, and, in the second, of *functions of two variables*. Similarly, four-termed functional relations are called *functions of three variables*, and so on. In designating functions with any number of arguments, it is customary to employ the variables "f", "g", ...; the formula:

$$z = f(x, y),$$

when the symbolism:

$$x = yRz$$

is employed, the relation R is usually referred to as an *operation*, or, more specifically, *a binary operation*. The four fundamental operations of addition, subtraction, multiplication, and division may serve as examples, and also such logical operations as union and intersection of classes or relations (see Sections 4.4 and 5.2). The content of the two concepts of a function of two variables and of a binary operation is evidently exactly the same. It should, perhaps, be noted that functions of one variable are sometimes also called operations, and, in particular, *unary operations*; in the calculus of classes, for instance, the forming of the complement of a class is usually thought of, not as a function, but as an operation.

Although the many-termed relations play an important part in various sciences, the general theory of these relations is yet in the initial stage; when speaking of a relation, or of the theory of relations, one usually has only two-termed relations in mind. A more detailed study has so far only been made of one particular category of three-termed relations, namely, a category of binary operations, as the prototype of which we may consider the ordinary arithmetical addition. These investigations are carried on within the framework of a special mathematical discipline known as the *Theory of Groups*. We shall get acquainted with certain concepts from the theory of groups – and thereby also with certain general properties of binary operations – in the second part of this book.

5.9 The Importance of Logic for Other Sciences

We have discussed the most important concepts of contemporary logic, and in doing so we have become acquainted with some laws (very few, by the way) concerning these concepts. It had not been our intention, however, to give a complete list of all logical concepts and laws of which one avails oneself within scientific arguments. This, incidentally, is not necessary, as far as the study and promotion of other sciences are concerned, even of mathematics which is especially closely related to logic. Logic is justly considered the basis of all other sciences, if only for the reason that in every argument we employ concepts taken from the field of logic and that every correct inference proceeds in accordance with the laws of that discipline. But this does not imply that a thorough knowledge of logic is a necessary condition for correct thinking; even professional mathematicians, who in general, do not commit errors in their inferences, usually do not know logic to such an extent as to be conscious of all logical laws of which they make unconscious use. All the same, there can be no doubt that the knowledge of logic is of considerable practical importance for everyone who desires to think and infer correctly, since it enhances the innate and acquired facilities to this effect and, in particularly critical cases, prevents the committing of mistakes. As far as, in particular, the construction of mathematical theories is concerned, logic plays a part of far-reaching importance also from the theoretical point of view; this problem will be discussed in the next chapter.

Exercises

1. Consider the relation of being father, that is to say, the relation expressed by the sentential function:

$$xFy \overset{def}{\longleftrightarrow} x \text{ is the Father of } y.$$

Do all human beings belong to the domain of this relation? And do they all belong to the counter-domain?

2. Consider the following seven following relations among people:

$$xFy \overset{def}{\longleftrightarrow} x \text{ is the Father of } y$$
$$xMy \overset{def}{\longleftrightarrow} x \text{ is the Mother of } y$$
$$xCy \overset{def}{\longleftrightarrow} x \text{ is the Child of } y$$
$$xBy \overset{def}{\longleftrightarrow} x \text{ is the Brother of } y$$
$$xSy \overset{def}{\longleftrightarrow} x \text{ is the Sister of } y$$
$$xHy \overset{def}{\longleftrightarrow} x \text{ is the Husband of } y$$
$$xWy \overset{def}{\longleftrightarrow} x \text{ is the Wife of } y$$

Find, if possible, simple names for the following relations:

(a) $\breve{B \cup S}$ (b) $H \cup W$ (c) $C/H \cup W/C$ (d) F/M

(e) $\breve{M}/\breve{S}$ (f) $F/(H \cup W)$ (g) $M/\breve{C}$ (h) $B/\breve{C}$

3. Explain the meanings of the following formulas, and determine which of them are true:

(a) $F \subseteq M'$ (b) $\breve{B} = S$ (c) $F \cup M = \breve{C}$

(d) $H/M = F$ (e) $B/S \subseteq B$ (f) $S \subseteq C/\breve{C}$

4. Show by means of an example that the relation on relations $R/S = S/R$ is not always satisfied.

5. Which among the properties of relations discussed in Section 5.3 are possessed by the following relations:

(a) The relation of divisibility in the set of natural numbers.
(b) The relation of being relatively prime in the set of natural numbers.
(c) The relation of congruence in the set of polygons.
(d) The relation of being longer in the set of line segments.
(e) The relation of being perpendicular in the set of lines in the plane.
(f) The relation of intersecting in the set of geometric configurations.
(g) The relation of simultaneity in the class of physical events.
(h) The relation of temporally preceding in the class of physical events.
(i) The relation of being related in the class of human beings.
(j) The relation of fatherhood in the class of human beings.

7. Consider the following new definition:

Definition 5.25. A relation R is said to be *intransitive* if it satisfies:

$$R \text{ is intransitive } \overset{def}{\longleftrightarrow} \forall x, y, z \in K \ ((xRy \wedge yRz) \rightarrow \sim xRz).$$

Which of the relations from Exercise 5 are intransitive?

6. Given a point in the plane, consider the set of all disks in that plane with the given point as their common center. Show that this set is ordered by the relation of "$\subset$". What if the disks were not concentric? Would this still be an ordering if the disks did not lie in the same plane?

7. Consider an arbitrary relation R and its negation R' given a fixed class K. Show that the following statements of the theory of relations are true:
(a) R reflexive $\rightarrow R'$ is irreflexive.
(b) R symmetrical $\rightarrow R'$ is symmetrical.

8. Show that, if the relation R has each one of the properties discussed in Section 5.3, the converse relation $\breve{R}$ possesses the same property. (Note: There will be 7 different proofs).

9. The properties of relations which were introduced in Section 5.3 can easily be expressed in terms of the calculus of relations, provided the class K to which they refer is the universal class.
(a) The formulas:

$$R/R \subseteq R \text{ and } D \subseteq R \cup \breve{R},$$

for instance, express that the relation R is transitive and connected, respectively. Explain why; recall the meaning of the symbol "D" from Section 5.2.
(b) Express similarly that the relation R is symmetrical, asymmetrical, or intransitive.

10. Which of the relations expressed by the following formulas are functions:
(a) $2x + 3y = 12$
(b) $x^2 = y^2$
(c) $x + 2 > y - 3$
(d) $y + x = x^2$

11. Consider the function expressed by the formula:

$$y = x^2 + 1.$$

State the domain and range.

12. Consider the function expressed by the formula:

$$y = 3x + 1.$$

Show that this function is a bijection from the interval $[0, 1]$ to the interval $[1, 4]$. What conclusion may be drawn from this concerning the cardinal number of those intervals?

13. Consider the function expressed by the formula:

$$y = 2^x.$$

Using this function, show that the set of all numbers is equinumerous with the set of all positive numbers.

14. Which of the three-termed relations expressed by the following formulas are functions:
(a) $x + y + z = 0$
(b) $x \cdot y > z$
(c) $x^2 + y^2 = z^2$
(d) $z + 2 = x^2 + y^2$

Notes

[1]De Morgan and Peirce (cf. endnotes 2 of Chapter 1 and 5 of Chapter 2) were the first to develop the theory of relations, especially that part of it known as the calculus of relations (cf. Section 5.2). Their work was systematically expanded and completed by the German logician E. Schröder (1841-1902). Schröder's *Algebra und Logik der Relative* (Leipzig 1895), which appeared as the third volume of his comprehensive work *Vorlesungen über die Algebra der Logik.* is still the only exhaustive account of the calculus of relations.

[2]The first to call to attention to the property of infinite classes discussed here was the German philosopher and mathematician B. Bolzano (1781-1848) in his book *Paradoxien des Unendlichen* (Leipzig 1851, posthumously published); in his work we already find the first beginnings of the contemporary theory of sets. The above property was later employed by Peirce (cf. endnote 2 of Chapter 1) and others in order to formulate an exact definition of a finite and of an infinite class.

Part II

The Deductive Method and Formal Systems

Chapter 6

On The Deductive Method

6.1 Primitive and Defined Terms, Axioms and Theorems

We shall now attempt an exposition on the fundamental principles that are to be applied in the construction of logic and mathematics. The detailed analysis and critical evaluation of these principles are tasks of a special discipline, called the *Methodology of the Deductive Sciences* or the *Methodology of Mathematics*. For anyone who intends to study or advance some science it is undoubtedly important to be conscious of the method which is employed in the construction of that science. We shall see that, in the case of mathematics, the knowledge of that method is of particularly far-reaching importance, for lacking such knowledge it is impossible to comprehend the nature of mathematics.

The principles with which we shall get acquainted serve the purpose of securing for the knowledge acquired in logic and mathematics the highest possible degree of clarity and certainty. From this point of view a method of procedure would be ideal, if it permits us to explain the meaning of every expression occurring in this science and to justify each of its assertions. It is easy to see that this ideal can never be realized. In fact, when one tries to explain the meaning of an expression, one uses, of necessity, other expressions; and in order to explain, in turn, the meaning of these expressions, without entering into a vicious circle, one has to resort to further expressions again, and so on. We thus have the beginning of a process which can never be brought to an end, a process which, figuratively speaking, may be characterized as an *infinite regress* (a *regressus in infinitum*). The situation is quite analogous as far as the justification of the asserted statements of the science is concerned; for, in order to establish the validity of a statement, it is necessary to refer back to other statements, and (if no vicious cycle is to occur) this leads again to an infinite regress.

Logic, New New Edition.
Karl Frinkle.

By way of a compromise between that unattainable ideal and the realizable possibilities, certain principles concerning the construction of mathematical disciplines have emerged that may be described as follows.

When we set out to construct a given discipline, we distinguish, first of all, a certain small group of expressions with the following property:

Definition 6.1. Terms of a specific discipline that seem to us to be immediately understandable are called *primitive terms* or *undefined terms*, and we employ them without explaining their meanings.

At the same time we adopt the principle: not to employ any of the other expressions of the discipline under consideration, unless its meaning has first been determined with the help of primitive terms and of such expressions of the discipline whose meanings have been explained previously.

Definition 6.2. A sentence which determines the meaning of a term using only primitive terms is called a *definition.*

Definition 6.3. A new term introduced in a definition using only primitive is known as a *defined term.*

We proceed similarly with respect to the asserted statements of the discipline under consideration. Some of these statements, which to us have the appearance of evidence, are chosen as *axioms*.

Definition 6.4. An *axiom* is a statement which we accept as true without in any way establishing its validity.

On the other hand, we agree to accept any other statement as true only if we have succeeded in establishing its validity, and to use, while doing so, nothing but axioms, definitions, and such statements of the discipline the validity of which has already been established previously.

Definition 6.5. A statement which is established in the method just described, is called a *proven statement*, or *theorem.*

Definition 6.6. The process of establishing the validity of a theorem is called a *proof.*

Definition 6.7. A *deduction* is the establishment, within logic or mathematics, of a statement on the basis of other statements. The statement established is said to be *derived* from the other statements.

Contemporary mathematical logic is one of those disciplines which are constructed in accordance with the principles just stated; unfortunately, it has not been possible within the narrow framework of this book to give this important fact due prominence. If any other discipline is constructed according to these principles, it is already based upon logic; logic, so to speak, is then already presupposed. This means that all expressions and laws of logic are treated on an equal footing with the primitive terms and axioms of the discipline under construction; the logical terms are used in the formulation of the axioms, theorems, and definitions, for instance, without an explanation of their meaning, and the logical laws are applied in proofs without first establishing their validity. Sometimes it is even convenient not only to use logic in the construction of a discipline but to presuppose in the same sense certain mathematical disciplines previously constructed. Thus logic itself does not presuppose any preceding discipline. In the construction of arithmetic as a special mathematical discipline logic is presupposed as the only preceding discipline; on the other hand, in the case of geometry it is expedient, though not unavoidable, to presuppose not only logic but also arithmetic.

With reference to the last remarks, it is necessary to make certain corrections in the formulation of the principles stated above. Before undertaking the construction of a discipline, those disciplines have to be enumerated that are to precede the given discipline; all requirements concerning the defining of expressions and the proving of statements, however, are limited to those expressions and statements which are specific for the discipline under construction, that is those which do not belong to the preceding disciplines.

Definition 6.8. The method of constructing a discipline in strict accordance with the principles laid down above is known as the *deductive method*; and the disciplines constructed in this manner are called *deductive theories*.[1]

The view has become more and more common that the deductive method is the only essential feature by means of which the mathematical disciplines can be distinguished from all other sciences; not only is every mathematical discipline a deductive theory, but also, conversely, every deductive theory is a mathematical discipline (according to this view deductive logic is also to be counted among the mathematical disciplines). We will not enter here into a discussion of the reasons in favor of this view, but merely remark that it is possible to put forward ponderable arguments in its support.

6.2 Model and Interpretation of a Deductive Theory

As a result of a consistent application of the principles presented in the preceding section, deductive theories acquire certain interesting and important features which we shall describe here. Since the questions which we are going to discuss have a rather involved and abstract character, we shall try to elucidate them by means of a concrete example.

Suppose we are interested in general facts about the congruence of line segments, and we intend to build up this fragment of geometry as a special deductive theory. We accordingly stipulate that the variables "x", "y", "z", ... denote segments. As primitive terms, we choose the symbols "S" and "$\cong$". The former is an abbreviation for "*the set of all line segments*"; the latter designates the relation of congruence.

Axiom I. $\forall x \in S\ (x \cong x)$

Axiom II. $\forall x, y, z\ \in S\ \ ((x \cong z \wedge y \cong z) \to x \cong y)$

Axiom I gives the definition of the relation of congruence being reflexive. Axiom II states that two segments congruent to the same segment are congruent to each other. Various theorems on the congruence of segments may be derived from these axioms. For instance, we will introduce two theorems along with their proofs:

Theorem I. $\forall y, z \in S\ (y \cong z \to z \cong y)$

Proof. We will prove this theorem by manipulating Axioms I and II and deriving the theorem exactly.

(1)	$(z \cong z \wedge y \cong z) \to z \cong y$	Instance of Axiom II, $z : x$
(2)	$z \cong z$	Instance of Axiom I, $x : z$
(3)	$(z \cong z \wedge y \cong z) \leftrightarrow y \cong z$	Instance of: $T \wedge p \leftrightarrow p$, $T : z \cong z$, $p : y \cong z$
(4)	$y \cong z \to z \cong y$	Substitute (3) into (1)

□

Theorem II. $\forall x, y, z \in S\ \ ((x \cong y \wedge y \cong z) \to x \cong z)$

Proof. We will proceed differently than we did in the proof of Theorem I. Here, we will assume the hypothesis, which is $x \cong y \wedge y \cong z$, and we will use Axioms I and II along with Theorem I to prove that the conclusion must also be true. Note that since we have already proven Theorem I using only Axioms I and II, we can now use it in any future proofs involving theorems based on the primitive terms and axioms described.

(1)	$x \cong y \wedge y \cong z$	Assume hypothesis
(2)	$(x \cong y \wedge y \cong z) \to x \cong y$	Law of And, Breaking Apart $p : x \cong y$, $q : y \cong z$
(3)	$(x \cong y \wedge y \cong z) \to y \cong z$	Law of And, Breaking Apart $q : x \cong y$, $p : y \cong z$

(4)	$x \cong y$	Rule of Detachment on (1) and (2)
(5)	$y \cong z$	Rule of Detachment on (1) and (3)
(6)	$y \cong z \to z \cong y$	Theorem I
(7)	$z \cong y$	Rule of Detachment on (5) and (6)
(8)	$x \cong y \wedge z \cong y$	Rule of And, Joining Together (4) and (7)
(9)	$(x \cong y \wedge z \cong y) \to x \cong z$	Instance of Axiom II, $y : z$, $z : y$
(10)	$x \cong z$	Rule of Detachment on (9) and (10)

Starting with the assumption $x \cong y \wedge y \cong z$, we have concluded, using exclusively the given axioms, primitive terms, the previously proven Theorem I, and the rules of logic, that $x \cong z$. □

In connection with these proofs we make the following remarks. Our miniature deductive theory rests upon a suitably selected system of primitive terms and axioms. Our knowledge of the things denoted by primitive terms, that is, of the segments and their congruence, is very comprehensive and is by no means exhausted by the adopted axioms. But this knowledge is, so to speak, our private concern which does not exert the least influence on the construction of our theory. In particular, in deriving theorems from the axioms, we make no use whatsoever of this knowledge, and behave as though we did not understand the content of the concepts involved in our considerations, and as if we knew nothing about them that had not been expressly asserted in the axioms. *We disregard, as it is commonly put, the meaning of the primitive terms adopted by us, and direct our attention exclusively to the form of the axioms in which these terms occur.*

This implies a very significant and interesting consequence. Let us replace the primitive terms in all axioms and theorems of our theory by suitable variables, for instance, the symbol "S" by the variable "K" denoting classes, and the symbol "$\cong$" by the variable "R" denoting relations (in order to simplify the considerations, we disregard here and theorems which contain defined terms). The statements of our theory will then be no longer sentences, but will become sentential functions which contain two free variables "K" and "R", and which expresses, in general, the fact that the relation R has this or that property in the class K (or, more precisely, that this or that relation holds between K and R; cf. Section 5.1). For instance, we can now rewrite Axiom I and Theorems I and II using relation properties previously defined in Section 5.3:

Axiom I′. *The relation R is reflexive in the class K.*

Theorem I′. *The relation R is symmetric in the class K.*

Theorem II′. *The relation R is transitive in the class K.*

Unfortunately, Axiom II does not define any property of a relation as found in Section 5.3. However, if we refer to the relation property defined in this axiom as '*pseudotransitivity*':

Definition 6.9. A relation R is *pseudotransitive* in the class K if for any three elements x, y, and z, the formulas xRz and yRz implies that xRy:

$$R \text{ is } \textit{pseudotransitive} \overset{def}{\longleftrightarrow} \forall\, x, y, z \in K \;\; ((xRz \wedge yRz) \rightarrow xRy)\,.$$

We then arrive at the following reformulation of Axiom II:

Axiom II′. *The relation R is pseudotransitive in the class K.*

Since, in the proofs of our theory, we make use of no properties of the class of segments and of the relation of congruence but those which were explicitly stated in the axioms, every proof can be considerably generalized, for it can be applied to any class K and any relation R having those properties. As a result of such generalization of the proofs, we can correlate with any theorem of our theory a general law belonging to the domain of logic, namely to the theory of relations, and stating that every relation R which is reflexive and pseudotransitive in the class K also has the property expressed in the theorem considered. So, for instance, the following two laws of the theory of relations correspond to Theorems I and II:

Theorem I″. *Every relation R which is reflexive and pseudotransitive in the class K is also symmetrical in the class K.*

Theorem II″. *Every relation R which is reflexive and pseudotransitive in the class K is also transitive in the class K.*

If a relation R is reflexive and pseudotransitive in a class K, we say that K and R together form a *model of the axiomatic system*. One model of the axiom system is formed, for instance, by the class of the segments and the relation of congruence, that is, the things denoted by the primitive terms; of course, this model also satisfies all the theorems deduced from the axioms.

Definition 6.10. Given a collection of axioms which are based solely on properties of classes and relations, relations R, S, T, ... and a class K are said to be a *model of the axiomatic system* if the relations and the class K satisfy the properties outlined in each of the axioms.

Note here that we have included the idea that more than one relation may be involved in the system of axioms, while we can always assume there is a single class K for which the axioms are applied to, for if there are two such classes, K_1 and K_2, we simply set $K = K_1 \cup K_2$.

We are able to exhibit many different models for our system consisting of Axioms I′ and II′. To obtain such a model, we select within any other deductive theory two specific constants, say "K" and "R" (the former denoting a class, the latter a relation), then we replace "S" by "K" and "$\cong$" by "R" everywhere in the system, and finally we show that the sentences thus obtained are theorems, or possible axioms, of the new theory. If we have succeeded in doing so, we say that we have found an *interpretation of the axiomatic system*. If we now replace the primitive terms "S" and "$\cong$" by "K" and "R", not only in the axioms, but also in the theorems of our theory, we can be sure in advance that all sentences thus obtained will be true sentences of the new deductive theory. Take, for instance, the proof of Theorem I. One can write this proof in terms of "K" and "R" only, thus any interpretation of the axiomatic system must satisfy the form of the proof:

Theorem I″. *Every relation R which is reflexive and pseudotransitive in the class K is also symmetrical in the class K.*

Proof.

(1)	$(zRz \wedge yRz) \rightarrow zRy$	Instance of Axiom II′, $z : x$
(2)	zRz	Instance of Axiom I′, $x : z$
(3)	$(zRz \wedge yRz) \leftrightarrow yRz$	Instance of: $T \wedge p \leftrightarrow p$, $T : zRz$, $p : yRz$
(4)	$yRz \rightarrow zRy$	Substitute (3) into (1)

□

We shall give here two concrete examples of interpretations of our miniature theory. Let us replace in Axioms I and II the symbol "S" by the symbol of the universal class "U", and the symbol "$\cong$" by the identity sign "$=$". This yields the axiomatic system:

Axiom Î. $\forall x \in U \ (x = x)$

Axiom IÎ. $\forall x, y, z \ \in U \ \left((x = z \wedge y = z) \rightarrow x = y \right)$

As can be seen immediately, the axioms will then become logical laws (in fact, Laws 3.2 and 3.5 of Section 3.2 in a slightly modified form). The universal class and the relation of identity constitute, therefore, a model of the axiom system, and our theory has found an interpretation within logic. This, if in Theorems I and II we replace the symbols "**S**" and "$\cong$" by the symbols "U" and "$=$", we are sure to arrive at true logical sentences (in fact, these are Laws 3.3 and 3.4 of Section 3.2).

Next, let us consider the set of all real numbers, denoting it by "$\mathbb{R}$", and we will call two real numbers x and y *equivalent*, in symbols:

$$x \equiv y,$$

if their difference $x - y$ is an integer. As examples:

$$\frac{15}{4} \equiv \frac{11}{4}, \quad \frac{3}{7} \equiv -\frac{4}{7}, \quad \frac{15}{4} \not\equiv \frac{9}{4}, \quad \frac{2}{7} \not\equiv \frac{4}{7}$$

If now, in both axioms, the primitive terms are replaced by "$\mathbb{R}$" and "$\equiv$", it can be easily shown that the resulting sentences are true theorems of arithmetic. Thus, our theory possesses an interpretation within arithmetic, for the set of numbers "$\mathbb{R}$" and the relation of equivalence "$\equiv$" constitute a model of the axiom system. And again, without any special reasoning we are sure that Theorems I′ and II′ will become true arithmetical statements if they are subjected to the same transformations as the axioms.

The general facts described above have many interesting applications in methodological researches. We shall illustrate this here by means of one example only; we shall show how it may be proved that certain sentences cannot be logically deduced from an axiomatic system. Let us consider the following conjecture, which is formulated in logical terms and in the primitive terms of our theory only:

Conjecture A. *There exists two elements $x, y \in \mathbb{S}$ for which $\sim x \cong y$. (In other words, there exists two segments which are not congruent).*

This sentence appears to be true, hence a great idea for a theorem. Nevertheless, no attempts to prove it based on Axioms I and II give a positive result. Thus the idea arises that Conjecture A cannot be deduced at all from our axioms. In order to confirm this, we argue in the following way: If Conjecture A could be proved on the basis of our axiomatic system, then, as we know, every model of this system would satisfy Conjecture A; if, therefore, we succeed in finding an interpretation of the axiomatic system for which the conjecture does not hold, we have then proven that the conjecture cannot be logically deduced from Axioms I and II. Now it turns out that producing such a model does not present any difficulties. Let us consider, for instance, the set $I = \{0, 1\}$ and the relation of equivalence $\cong$ between numbers which was discussed above. We already know from the preceding remarks that the set I and the relation $\cong$ constitute a model for our axiomatic system; Conjecture A however is not satisfied by this model of our axiom system, for there are no two elements which are not congruent. The type of reasoning just applied is known as the *method of proof by exhibiting a model*, or *method of proof by interpretation*. Another model which would have worked is using an arbitrary class K with the universal relation U, which holds between any two elements of the class K.

The facts and concepts discussed here can be related, without essential change, to other deductive theories. In the next section we shall try to describe them in quite a general way.

6.3 Law of Deduction

We consider any deductive theory based upon a system of primitive terms and axioms. In order to simplify our considerations, we assume that this theory presupposes logic only, that is, logic is the only theory preceding the given theory (cf. Section 6.1). Let us imagine that in all the statements of our theory the primitive terms are replaced by suitable variables throughout (as in Section 6.2, and again for the sake of simplicity, we disregard theorems containing defined terms). The statements of the theory considered become sentential functions containing as free variables those symbols by which the primitive terms had been replaced and not containing any constants but those belonging to logic. Given certain things one can find out whether they satisfy all the axioms of our theory, or, to be exact, the sentential functions obtained from these axioms in the manner just described. If it turns out that this is the case, the things under consideration form a model for the axiomatic system of our deductive theory. In a quite analogous manner we can find out whether, therefore, they form a model of this system (it is not excluded that the system consists of a single statement).

A model of the axiomatic system is formed, for instance, by those things which are denoted by primitive terms of the given theory, since we assume that all axioms are true sentences; this model satisfies, of course, all the theorems of our theory. But as far as the construction of our theory is concerned, this model takes no distinguished place among all other models. When deducing theorems from the axioms, we do not think of the specific properties of this model, and we make use of only those properties which are explicitly stated in the axioms and, therefore, belong to every model of the axiomatic system. Consequently, every proof of a particular theorem of our theory can be extended to every model of the axiomatic system and can be thus transformed into a much more general argument no longer belonging to our theory but to logic; and as a result of this generalization we obtain general logical statements (like Theorems I″ and I″) which establishes the fact that the theorem in question is satisfied by every model of our axiomatic system. The final conclusion at which we arrive in this way can be put in the following form:

Every theorem of a given deductive theory is satisfied by any model of the axiomatic system of this theory; and moreover, to every theorem there corresponds a general statement which can be formulated and proved within the framework of logic and which establishes the fact that the theorem in question is satisfied by any such model.

We have here a general law from the domain of the methodology of deductive sciences which, when formulated in a slightly more precise way, is known as the *Law of Deduction*.[2]

The tremendous practical importance of this law results from the fact that we are usually able to exhibit numerous models of the axiomatic system of a particular theory, even without leaving the field of the deductive sciences. In order to arrive at such a model it is sufficient to select certain constants from some other deductive theory (which can be logic or a theory presupposing logic), to put them in the axioms in place of the primitive terms, and to show that the sentences obtained in this way are asserted statements of that other theory. We say in this case that we have found an *interpretation of the axiomatic system of the original theory within the other theory.* (It may, in particular, occur that the constants chosen belong to the theory originally considered, in which case some of the primitive terms may even have remained unchanged; the given axiomatic system is then said to have found a *new interpretation within the theory under consideration.*) We shall also subject the theorems of the original theory to an analogous transformation, replacing the primitive terms throughout by those constants that had been employed in the interpretation of the axioms. On the basis of the law of deduction we can then be sure in advance that the sentences arrived at in this manner are asserted statements of a new theory. We can formulate this in the following way:

All theorems proved on the basis of a given axiomatic system remain valid for any interpretation of the system.

It is redundant to give a special proof for any of these transformed theorems; it would in any case be a task of purely mechanical nature, for it would be sufficient to transfer the corresponding argument from the field of the original theory and to subject it to the same transformations that had been carried out with respect to the axioms and theorems. Every proof within a deductive theory contains an unlimited number of other analogous proofs.

The facts described above demonstrate the great value of the deductive method from the point of view of economy of human thought. They are also of far-reaching theoretical importance, if only for the reason that they establish a foundation for various arguments and researches within the methodology of deductive sciences. In particular, the law of deduction is the theoretical basis for all so-called *proofs by interpretation*; we have already encountered one example of such proofs in the preceding section, and we shall meet with various other examples in the second part of the book.

For reasons of exactness it may be added that the considerations sketched here are applicable to any deductive theory in whose construction logic is presupposed, whereas their application to logic itself brings about certain difficulties which we would rather not discuss here. If a deductive theory presupposes not only logic, but also other theories, some of the formulations given above assume a somewhat more complicated form.

The common source of the methodological phenomena discussed here is the fact pointed out in the preceding section, namely that, in constructing a deductive theory, we disregard the meaning of the axioms and take into account only their form. It is for this reason that people, when referring to those phenomena, speak about the purely *formal character* of deductive sciences and of all reasonings within these sciences.

From time to time one finds statements which emphasize the formal character of mathematics in a paradoxical and exaggerated way; although fundamentally correct, these statements may become a source of obscurity and confusion. Thus one hears, and even reads occasionally, that no definite content may be ascribed to mathematical concepts; that in mathematics we do not really know what we are talking about, and that we are not interested in whether our assertions are true. One should approach such judgements rather critically. If, in the construction of a theory, one behaves as if one did not understand the meaning of the terms of this discipline, this is not at all the same as denying those terms any meaning. It is, admittedly, sometimes the case that we develop a deductive theory without ascribing a definite meaning to its primitive terms, thus dealing with the latter as with variables; in this case we say that we treat the theory as a *formal system*. But this is a comparatively rare situation (not even taken into account in our general characterization of deductive theories given in Section 6.1), and it occurs only if it is possible to give several interpretations for the axiomatic system of this theory, that is, if there are several ways available of ascribing concrete meanings to the terms occurring in the theory, but if we do not desire to give preference in advance to any one of these ways. A formal system, on the other hand, for which we are unable to give a single interpretation, would, presumably, be of interest to nobody.

In conclusion we shall call attention to certain interesting examples of interpretations of mathematical disciplines, which are much more important than those given in Section 6.2.

The axiomatic system of arithmetic may be interpreted within geometry: given an arbitrary straight line, it is possible to define such relations between its points and operations on its points which satisfy all the axioms, and hence also all the theorems, of arithmetic, which are concerned with corresponding relations between numbers and operations on numbers. (This is closely connected with a circumstance mentioned in Section 5.7, namely, the possibility of establishing a one-to-one correspondence between all the points of a line and all numbers.) Conversely, the axiomatic system of geometry also possesses an interpretation within arithmetic. The uses to which these two facts can be put are manifold. Geometrical configurations may, for instance, be employed in order to give a visual image of various facts in the field of arithmetic. A procedure known as the graphical method; on the other hand, it is possible to investigate geometrical facts with the help of arithmetical or algebraical methods. There is even a special branch of geometry, known as *analytic geometry*, which is concerned with all investigations of this type.

Arithmetic, as we have seen previously, may be built up as a part of logic (cf. Section 4.6). But if we treat arithmetic as an independent deductive theory, resting upon its own system of primitive terms and axioms, its relation to logic can be described as follows: arithmetic possesses an interpretation within logic (with the understanding that the Axiom of Infinity be included in logic, cf. Section 4.6). In other words, it is possible within logic to define the necessary concepts which satisfy all the axioms, and hence also all the theorems, of arithmetic. If we remember that geometry has an interpretation in arithmetic, we arrive at the conclusion that also geometry can be interpreted within logic. All these are facts which are exceedingly significant from the methodological point of view.

6.4 Selection of Axioms and Primitive Terms

We now turn to the discussion of a few problems of a more special nature, which, however, concern fundamental components of the deductive method, namely the choice of the primitive terms and axioms as well as the construction of definitions and proofs.

It is important to realize the fact that we have a large degree of freedom in the selection of the primitive terms and axioms; it would be quite erroneous to believe that certain expressions cannot be defined in any possible way, or that certain statements can, on principle, not be proved.

Definition 6.11. Two systems of sentences of a given theory are *equipollent* if each sentence of the first system can be derived from the sentences of the second, together with theorems of the preceding theories, and, conversely, if every sentence of the second system can be derived from the sentences of the first.

Obviously any sentences occurring in both systems do not have to be derived. Let us imagine, further, that a certain deductive theory has been built upon the basis of some axiomatic system, and that in the course of its construction we come across a system of statements equipollent in the sense just defined to the axiomatic system. A concrete example can be obtained in connection with the miniature theory of the congruence of segments discussed in Section 6.2: it is easy to show that its axiomatic system is equipollent to the system of sentences consisting of Axiom I with Theorems I and II. If this kind of situation arises, then, from the theoretical point of view, it would be possible to reconstruct the entire theory in such a manner that the statements of the new system are taken as axioms, while the former axioms are proved as theorems. Even the circumstance that the new axioms may, at first, to a much lesser degree have the appearance of immediate evidence is inessential;

for every sentence becomes evident to a certain degree, once it has been derived in a convincing manner from other evident sentences. All this applies likewise, *mutalis mutandis*, to the primitive terms of a deductive theory; the system of these terms may be replaced by any other system of terms of the theory in question, provided only the two systems are equipollent in the sense that each term of the first system can be defined by means of terms of the second together with terms taken from the preceding theories, and vice versa. It is not for theoretical reasons (or, at least, not only for theoretical reasons) that we decide to select a certain system of primitive terms and axioms in preference to any of the other possible equipollent systems; other factors play a role here: practical, didactical, even esthetic ones. Sometimes it is a question of choosing the simplest possible primitive terms and axioms, then again it may be desirable to get along with as few of them as possible, or we may prefer such primitive terms and axioms as would enable us, in the simplest possible way, to define those terms and to prove those statements of a given theory in which we are especially interested.

Another problem arises in close connection with these remarks. Fundamentally, we strive to arrive at an axiomatic system which does not contain a single superfluous statement, that is, a statement which can be derived from the remaining axioms and which, therefore, might be counted among the theorems of the theory under construction.

Definition 6.12. A system of axioms is called *independent* (or a *system of mutually independent axioms*) if no axiom in the system can be determined to be superfluous.

In a similar fashion, we can determine whether or not a system of primitive terms is independent. Often, however, one does not insist on these methodological postulates for practical, didactical reasons, particularly in cases where the omission of a superfluous axiom or primitive term would bring about great complications in the construction of the theory.

6.5 Formalization of Definitions and Proofs

The deductive method is justifiably considered the most perfect of all methods employed in the construction of sciences. It disposes to a large extent of the possibility of obscurities and errors, without resorting to an infinite regress; and it is due to this method that any reasons for doubt as to the content of concepts or the truth of assertions of a given theory are considerably reduced and may hold at most for the few primitive terms and axioms.

One reservation has to be added to this statement however. The application of the deductive method will give the desired results only if all the definitions and proofs fulfil their tasks completely, that is, if the definitions make fully clear the meaning of all terms to be defined and if the proofs convince us wholly of the validity of all the theorems to be proved. It is far from easy to examine whether the definitions and proofs actually comply with these requirements; it is quite possible, for instance, that an argument which seems entirely convincing to one person is not even comprehensible to another. In order to remove any cause for doubt in this respect, the present-day methodology endeavours to replace subjective valuations in the examination of definitions and proofs by criteria of an objective nature, and to make the decision as to the correctness of definitions or proofs dependent exclusively upon their structure, that is, their exterior form. For this purpose, special *rules of definition* and *rules of proof* are stated. The first tell us what form the sentences should have which are used as definitions in the theory under consideration, and the second describe the kind of transformations to which statements of this theory may be subjected in order to derive other statements from them; each definition has to be laid down in accordance with the rules of definition, and each proof must be complete, that is, it must consist in a successive application of rules of proof to sentences previously recognized as true (cf. Sections 2.6 and 3.1). These new methodological postulates may be denoted as postulates of the *Formalization of Definitions and Proofs*; a discipline constructed in accordance with these new postulates is called a *Formalized Deductive Theory*.[3]

In the light of modern requirements, logic becomes the basis of the mathematical sciences in a much more thorough sense than it used to be. We may no longer be satisfied with the conviction that, due to our innate or acquired capacity for correct thinking, our argumentations are in accordance with the rules of logic. In order to give a complete proof of a theorem it is necessary to apply the transformations prescribed by the rules of proof not only to the statements of the theory with which we are concerned, but also to those of logic (and other preceding theories); and for this purpose we have to have a complete list of all logical laws at our disposal that are applied in the proofs. (See, for instance, the list of logical laws at the beginning of this book, which has hopefully proven useful up to this point.)

It is only by virtue of the development of deductive logic that, theoretically at least, we are today in a position to present every mathematical discipline in formalized form. In practice, however, this still involves considerable complications; a gain in exactitude and methodological correctness is accompanied by a loss in clarity and intelligibility. It would be far from sensible to demand that the proofs of theorems in an ordinary textbook of some mathematical discipline be given in complete form; one should, however, expect the author of a textbook to be intuitively certain that all their proofs can be brought into that form, and even to carry their considerations to the point from which a reader who has some practice in deductive thinking and sufficient knowledge of contemporary logic would be able to fill the remaining gaps without much difficulty.

6.6 Consistency and Completeness of a Deductive Theory

We shall now consider two methodological concepts which are of great importance from the theoretical point of view, while in practical respects they are of little significance. They are the concepts of *consistency* and of *completeness*.

Definition 6.13. A deductive theory is called *consistent*, or *non-contradictory*, if not two asserted statements of this theory contradict each other. In other words, if any two contradictory sentences, p and $\sim p$, (cf. Section 2.2) at least one cannot be proved.

Definition 6.14. A deductive theory is called *complete* if at least one of any pair of contradictory sentences, p and $\sim p$, formulated exclusively in the terms of the theory under consideration can be proven in this theory.

Definition 6.15. A sentence which has the property that its negation can be proven in a given theory is said to be *disproved*.

With this terminology, we can say that *a deductive theory is consistent if no sentences can be both proved and disproved in it*; on the other hand, *a theory is complete if every sentence formulated in the terms of this theory can be proved or disproved in it.* Both terms “consistent” and “complete” are applied, not only to the theory itself, but also to the axiomatic system upon which it is based.

Let us now try to get a clear idea of the importance of these two notions. Every discipline, even one constructed entirely correctly in every methodological respect, loses its value in our eyes if we have reason to suspect that not all assertions of this discipline are true. On the other hand, the value of a discipline should be considered greater, the larger the number of true sentences whose validity can be established in it. From this point of view, a discipline might be considered ideal if it contains among its asserted statements all true sentences which are relevant to that theory, and not a single false one. A sentence is here considered relevant if it is formulated entirely in terms of the discipline under consideration (and its preceding disciplines); after all, it cannot be expected that, say, in arithmetic all true sentences can be proved, even such as contain concepts of chemistry or biology.

Let us now imagine that a deductive theory is inconsistent, that is to say, that two contradictory sentences occur among its axioms and theorems; from a well-known logical law, namely Law 2.7, the Law of Contradiction (cf. Section 2.8), it follows that one of these sentences must be false. If, on the other hand, we assume the theory to be *incomplete*, there exist two relevant contradictory sentences of which neither can be proved in that discipline; and yet, by another logical law, i.e., Law 2.8, the Law of Excluded Middle, one of the two sentences must be true. We see from this that a deductive theory certainly falls short of our ideal unless it is both consistent and complete. (Thereby we do not mean to imply that every consistent and complete discipline must, *ipso facto*, be a realization of our ideal, that is, that it must contain among its asserted statements all true sentences and only such sentences.)

There is yet another aspect to the whole question which we have been considering. The development of any deductive science consists in formulating in the terms of this science problems of the type "*is such and such the case?*" and then attempting to decide these problems on the basis of these axioms that have been assumed. Any problems of this type may clearly be decided in one of two possible ways: in the affirmative or in the negative. On the first alternative, the answer runs: "*such and such is the case*"; and on the second: "*such and such is not the case*". The consistency and the completeness of the axiomatic system of a deductive theory now give us a guarantee that every problem of the kind mentioned can actually be decided within the theory, and moreover decided in one way only; the consistency excludes the possibility that any problem may be decided in two ways, that is, both affirmatively and negatively, and the completeness assures us that it can be decided in at least one way.

Closely connected with the problem of completeness is another, more general, problem which concerns incomplete as well as complete theories. It is the problem which consists in finding, for the given deductive theory, a general method which would enable us to decide whether or not any particular sentence formulated in the terms of this theory can be proved within this theory. This important problem is known as the *decision problem*.[4]

There are only a few deductive theories known of which it has been possible to show that they are consistent and complete. They are, as a rule, elementary theories of a simple logical structure and a modest stock of concepts. An example is given by sentential calculus, which has been discussed in Chapter 2, provided that it is considered as an independent theory and not as a part of logic (however, in applying the term "complete" to this theory, it is to be used in a slightly modified meaning). Perhaps the most interesting example of a consistent and complete theory is that supplied by elementary geometry; we have here in mind geometry limited to those confines wherein it has for centuries been taught in schools as a part of elementary mathematics, that is to say, a discipline in which the properties of various special kinds of geometrical figures such as lines, planes, triangles, circles are investigated, but in which the general concept of a geometrical configuration (a point set) does not occur.[5] The situation changes essentially as soon as one goes over to such sciences as arithmetic or advanced geometry. Probably no one working in these fields doubts their consistency; and yet, as has resulted from the latest methodological investigations, a strict proof of their consistency meets with great difficulties of a fundamental nature. The situation in regard to the problem of completeness is even worse:

Arithmetic and advanced geometry are incomplete.

It is possible to set up problems of a purely arithmetical or geometrical character that can be neither positively nor negatively decided within these disciplines. It might be supposed that this fact is merely an outcome of the imperfection of the axiomatic system and methods of proof at our disposal up to date, and that a suitable modification (for instance, an extension of the axiomatic system) may, in the future, yield complete systems. Deeper investigations, however, have shown this conjecture to be erroneous:

Never will it be possible to build up a consistent and complete deductive theory containing as its theorems all true sentences of arithmetic or of advanced geometry.

Moreover, it turns out that the decision problem likewise does not admit of a positive solution with respect to these disciplines; it is impossible to set up a general method which would allow us to differentiate between those sentences which can be proved within these disciplines and those which cannot be proved. All these results can be extended to many other deductive theories, and, in particular, to all those which either presuppose the arithmetic of integers (i.e., the theory of the four basic arithmetical operations on integers) or contain sufficient devices to develop this theory. Thus, for instance, these results can be applied to the general theory of classes, as follows from the remarks at the end of Section 4.6.[6]

In view of these last remarks, it is understandable that the concepts of consistency and completeness, in spite of their theoretical importance, exert little influence in practice upon the construction of deductive theories.

6.7 Widened Conception of the Methodology of Deductive Sciences

The investigations concerning consistency and completeness were among the most important factors which contributed to a considerable extension of the domain of methodological studies, and caused even a fundamental change in the whole character of the methodology of deductive science. That conception of methodology which was indicated at the beginning of the present chapter has, during the historical development of the subject, turned out to be too narrow. The analysis and critical evaluation of methods applied in practice in the construction of deductive sciences ceased to be the exclusive, or even main, task of methodology. The methodology of the deductive sciences became a general science of deductive sciences in an analogous sense as arithmetic is the science of numbers and geometry is the science of geometrical configurations. In contemporary methodology we investigate deductive theories as wholes as well as the sentences which constitute them; we consider the symbols and expressions of which these sentences are composed, properties and sets of expressions and sentences, relations holding among them (such as the consequence relation) and even relations between expressions and the things which the expressions "talk about" (such as the relation of designation); we establish general laws concerning these concepts.

In connection with the evolution through which methodology has passed there has arisen a need for applying new, more subtle and more precise methods of inquiry in this field. Methodology has become like those sciences which constitute its own subject matter, it has assumed the form of a deductive discipline. In view of the extended domain of investigations, the expression "the methodology of the deductive sciences" itself has ceased to appear appropriate enough; indeed, "methodology" means merely "the science of method". Consequently this expression is now often replaced by others – for instance, by a (not altogether happy) term "*theory of proof*", or by a (much better) term "*meta-logic and meta-mathematics*", which means about the same as "the science of logic and mathematics". Still another term has been recently coming into use, "*logical syntax and semantics of deductive sciences*", which stresses the analogy between the methodology of the deductive sciences and the grammar of everyday language.[7]

Exercises

1. The calculus of classes which was considered in Chapter 4 can be constructed as a separate deductive theory, presupposing sentential calculus only. In this construction we shall consider the symbols "U", "$\emptyset$", "$\subseteq$", and all operation signs introduced in Section 4.5 as primitive terms. We assume, further, the following nine axioms:[8]

Axiom I. $K \subseteq K$

Axiom II. $(K \subseteq L \wedge L \subseteq M) \to K \subseteq M$

Axiom III. $(K \cup L \subseteq M) \leftrightarrow (K \subseteq M \wedge L \subseteq M)$

Axiom IV. $(M \subseteq K \cap L) \leftrightarrow (M \subseteq K \wedge M \subseteq L)$

Axiom V. $K \cap (L \cup M) \subseteq (K \cap L) \cup (K \cap M)$

Axiom VI. $K \subseteq U$

Axiom VII. $\emptyset \subseteq K$

Axiom VIII. $U \subseteq K \cup K'$

Axiom IX. $K \cap K' \subseteq \emptyset$

From these axioms we may derive various theorems. Prove, in particular, the following theorems. Make use of the hints following each one, they help!

Theorem I. $K \cup K \subseteq K$

Hint: In Axiom III, perform the replacements $L : K$ and $M : K$. Notice that the right side of the equivalence thus obtained is satisfied by any class K (Axiom I).

Theorem II. $K \subseteq K \cap K$

Hint: The proof based upon Axioms IV and I is analogous to that of Theorem I.

Theorem III. $K \subseteq K \cup L \wedge L \subseteq K \cup L$

Hint: In Axiom III perform the substitution $M : K \cup L$; notice that the left side of the equivalence is an instance of Axiom I.

Theorem IV. $K \cap L \subseteq K \wedge K \cap L \subseteq L$

Hint: The proof is analogous to that of Theorem III.

Theorem V. $K \cup L \subseteq L \cup K$

Hint: In Axiom III, preform the substitution $M : L \cup K$, and compare the right side of the equivalence thus obtained with an instance of Theorem III where K and L are switched.

Theorem VI. $K \cap L \subseteq L \cap K$

Hint: The proof based upon Axiom IV and Theorem IV is analogous to that of Theorem V.

Theorem VII. $L \subseteq M \rightarrow K \cup L \subseteq K \cup M$

Hint: Assume the hypothesis of the theorem is satisfied, derive the formulas:

$$K \subseteq K \cup M, \quad L \subseteq K \cup M$$

The first of these formulas follows directly from Theorem III, and the second can be deduced from the hypothesis and Theorem III by Axiom II. Finally, apply Axiom III to these formulas.

Theorem VIII. $L \subseteq M \rightarrow K \cap L \subseteq K \cap M$

Hint: The proof is similar to that of the preceding theorem.

Theorem IX. $K \cap L \subseteq K \cap (L \cup M) \wedge K \cap M \subseteq K \cap (L \cup M)$

Hint: In Theorem III perform the substitution $K : L$ and $L : M$; to the formulas thus obtained apply Theorem VIII.

Theorem X. $(K \cap L) \cup (K \cap M) \subseteq K \cap (L \cup M)$

Hint: This theorem can be derived from Axiom III and Theorem IX.

Axioms III and IV, which play the most important role in the proofs of the above theorems, are called *Laws of Composition for Union and Intersection*, respectively.

2. In addition to Theorems I and II, the following theorems can be derived from the axioms of Section 6.2:

Theorem III. $\forall x, y, z \in S \ \ ((x \cong y \wedge x \cong z) \to y \cong z)$

Theorem IV. $\forall x, y, z \in S \ \ ((x \cong y \wedge y \cong z) \to z \cong x)$

Theorem V. $\forall x, y, z, t \in S \ \ ((x \cong y \wedge y \cong z \wedge z \cong t) \to x \cong t)$

Give a strict proof that the following systems of sentences are equipollent, in the sense established in Section 6.4, to the system consisting of Axioms I and II (and that each might, therefore, be chosen as a new axiomatic system). We reproduce Axioms I and II, along with Theorems I and II for sake of completeness.

Axiom I. $\forall x \in S \ (x \cong x)$

Axiom II. $\forall x, y, z \ \in S \ \ ((x \cong z \wedge y \cong z) \to x \cong y)$

Theorem I. $\forall y, z \in S \ (y \cong z \to z \cong y)$

Theorem II. $\forall x, y, z \in S \ \ ((x \cong y \wedge y \cong z) \to x \cong z)$

(a) The system consisting of Axiom I and Theorems I and II;
(b) The system consisting of Axiom I and Theorem III;

3. Along the lines of the remarks made in Section 6.2 formulate general laws of the theory of relations that represent a generalization of the results obtained in the preceding exercise.

Hint: These laws may, for instance, be given the form of equivalences, beginning with the words:

For a relation R to be reflexive and pseudotransitive in a class K, it is necessary and sufficient that ...

4. Consider the system of sentences (a) of Exercise 3. Exhibit models satisfying:
(a) The first two sentences of the system, but not the last.
(b) The first and third sentence, but not the second.
(c) The last two sentences, but not the first.

What conclusion may be drawn from the fact of the existence of such models with respect to the possibility of deriving any one of the three axioms from the others? Are these sentences mutually independent? (Cf. Sections 6.2 and 6.4.)

5. There exists a method of constructing sentential calculus as a formalized deductive theory which complies entirely with all the principles presented in Sections 6.1 and 6.5.[9] We may, for instance, assume the symbols "$\rightarrow$", "$\leftrightarrow$", and "$\sim$" (cf. Section 2.8) as primitive terms and the following seven sentences as axioms of sentential calculus:

Axiom I. $p \rightarrow (q \rightarrow p)$

Axiom II. $[\,p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$

Axiom III. $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

Axiom IV. $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$

Axiom V. $(p \leftrightarrow q) \rightarrow (q \rightarrow p)$

Axiom VI. $(p \rightarrow q) \rightarrow [(q \rightarrow p) \rightarrow (p \leftrightarrow q)]$

Axiom VII. $[(\sim q) \rightarrow (\sim p)] \rightarrow (p \rightarrow q)$

Furthermore, we agree to apply in proofs only one rule of inference, namely the Rule of Detachment. Furthermore, we are allowed to use instances of axioms and theorems (although for theorems, only after they have been proven). With the help of the Rule of Detachment and using instances of axioms, we are now in a position to deduce various theorems from our axioms. Give, in particular, complete proofs for the following theorems, making use of the hints which follow them:

Theorem I. $p \rightarrow p$

Hint: Substitute $q : p$ in Axioms I and II; notice that the first sentence thus obtained coincides with the antecedent of the second, and accordingly apply the Rule of Detachment.

Theorem II. $p \rightarrow [\,(p \rightarrow q) \rightarrow [(p \rightarrow q) \rightarrow q]\,]$

Hint: In Axiom I, substitute $q : (p \rightarrow q)$; in Axiom III, $p : (p \rightarrow q)$, $q : p$, and $r : q$. Notice that the consequent of the first implication thus obtained coincides with the antecedent of the second. Now replace, in Axiom III, q by the antecedent of the second implication and r by its consequent (leaving p, which is the antecedent of the first implication, unchanged). Then apply the Rule of Detachment twice. This proof is a typical instance of reasoning based upon Axiom III, which is another form of Law 2.5, the Law of Hypothetical Syllogism (cf Section 2.7).

Theorem III. $p \to [(p \to q) \to q]$

Hint: The proof is analogous to that of Theorem II. From Axiom II, derive the sentence:

$$[(p \to q) \to [(p \to q) \to q]\,] \to [(p \to q) \to q]$$

Compare the antecedent of this sentence with the consequent of Theorem II; accordingly, find a suitable instance of Axiom III and apply the Rule of Detachment twice.

Theorem IV. $[\,p \to (q \to r)] \to [q \to (p \to r)\,]$

Hint: From an instance of Axiom III derive the sentence:

$$[\,p \to (q \to r)\,] \to [\,[\,(q \to r) \to r] \to (p \to r)\,] \tag{6.1}$$

Furthermore, in Axiom III perform the instace of $p : q$, $q : [\,(q \to r) \to r]$, and $r : (p \to r)$. Notice that the antecedent of the implication thus derived can be obtained by an instance Theorem III. Perform this instance and, by applying the Rule of Detachment, derive:

$$[\,[\,(q \to r) \to r] \to (p \to r)\,] \to [q \to (p \to r)\,] \tag{6.2}$$

Notice now that the consequent of (6.1) is the same as the antecedent of (6.2); and accordingly, proceed as in the proof of Theorem II (by applying Axiom III again). Theorem IV is call the *Law of Commutation*.

Theorem V. $(\sim p) \to (p \to q)$

Hint: From an instance of Axiom I derive:

$$(\sim p) \to [\,(\sim q) \to (\sim p)\,]$$

Notice that the consequent of this sentence coincides with the antecedent of one of the axioms; and proceed as in the proof of Theorem II.

Theorem VI. $p \to [\,(\sim p) \to q\,]$

Hint: Find an instance of Theorem IV such that the antecedent of the resulting implication will be Theorem V, and then apply the Rule of Detachment. We have here a typical instance of reasoning based upon the Law of Commutation.

Theorem VII. $[\sim (\sim p)\,] \to (q \to p)$

Hint: The proof is analogous to that of Theorem II. From Theorem V and Axiom VII derive the sentences:

$$[\sim (\sim p)\,] \to [\,(\sim p) \to (\sim q)], \quad [\,(\sim p) \to (\sim q)\,] \to (q \to p)$$

Compare the antecedents and the consequents of these sentences.

Theorem VIII. $[\sim(\sim p)] \rightarrow p$

Hint: Reasoning as in the proof of Theorem VI, derive first from Theorems IV and VII, the sentence:

$$q \rightarrow [\,[\sim(\sim p)] \rightarrow p\,]$$

In this sentence put any one of our axioms in place of q, and apply the Rule of Detachment.

Theorem IX. $p \rightarrow [\sim(\sim p)]$

Hint: Find suitable instances of Axiom VII and Theorem VIII so as to be able to apply the Rule of Detachment.

Theorem X. $[\sim(\sim p)] \leftrightarrow p$

Hint: This theorem can be obtained from Axiom VI and Theorems VIII and IX by finding an instance of Axiom VI and then applying the Rule of Detachment twice.

Notes

[1]The deductive method cannot be considered an achievement of recent times. Already in the *Elements* of the Greek mathematician Euclid (about 300 B.C.) we find a presentation of geometry which leaves nothing much to be desired from the standpoint of the methodological principles stated above. For 2200 years, mathematicians have seen in Euclid's work the ideal and prototype of scientific exactitude. An essential progress in this field occurred only during the last 50 years, in the course of which the foundations of the basic mathematical disciplines of geometry and arithmetic were laid in accordance with all requirements of the present-day methodology of mathematics. Among the works to which we are indebted for this progress we will mention at least the following two, which have already become of historic importance: the collective *Formulaire de Mathématiques* (Torino 1895-1908) whose editor and main author was the Italian mathematician and logician G. Peano (1858-1932), and *Grundlagen der Geometrie* (Leipzig and Berlin 1899) by the great contemporary German mathematician D. Hilbert.

[2]The law was formulated by the author as a general methodological postulate, and was later proved exactly for various particular deductive theories.

[3]The first attempts to present the deductive theories in a formalized form are due to Frege who has already been quoted twice (cf. endnote 2 of Chapter 1). A very high level in the process of formalization was achieved in the works of the late Polish logician S. Leśniewski (1886-1939); one of his achievements is an exact and exhaustive formulation of the rules of definition.

[4]The import of the concepts and problems discussed in this section – and especially of the concepts of consistency and of the decision problem – was emphasized by Hilbert (c. endnote 1 of this chapter), who greatly stimulated many important investigations into the foundations of mathematics. Upon his instigation, these concepts and problems have of late been made the subject of intensive researches by a number of contemporary mathematicians and logicians.

[5]For the first proof of completeness of sentential calculus (and, thereby for the first positive result of the investigation concerned with completeness) we are indebted to the contemporary

American logician E.L. Post. The proof of the completeness of elementary geometry originates with the author.

[6]These exceedingly important achievements are due to the contemporary Austrian logician K. Gödel. His results concerning the decision problem were further extended by the contemporary American logician A. Church.

[7]Methodology of the deductive sciences in its widened meaning is a very young discipline. Its intensive development began only about twenty years ago – simultaneously (and, as it seems, independently) in two different centers: Göttingen under the influence of D. Hilbert (cf. footnote 1 of this chapter) and P. Bernays, and Warsaw where S. Leśniewski and J. Łukasiewicz among others, worked (cf. endnote 3 of this chapter and 2 of Chapter 1).

[8]The axiom system given here is essentially due to Schröder (cf. endnote 1 of Chapter 5). Various simple and interesting axiomatic systems for the calculus of classes were published by the contemporary American mathematician E. V. Huntington, to whom we are indebted for many important contributions concerning the axiomatic foundations of logical and mathematical theories.

[9]The method originates with Frege, whom we have already discussed in a previous note above.

Chapter 7

Propositional Languages and the Deductive Method

7.1 Formal Languages

In Chapter 6, we were introduced to the concept of the deductive theory, which encompassed such topics as primitive terms, axioms, models, as well as the concepts of completeness and consistency. We will next focus on the more logic-theoretical aspects of these topics, and will introduce a specific propositional system to expand upon the concepts aforementioned. A simple system was introduced in Exercise 10 of the previous chapter.

Definition 7.1. A *formal language* is a finite collection of symbols and a set of rules which can be applied to the symbols to form logical sentences, which can thus, by definition, be interpreted as either true or false.

We have been using a formal language throughout most of the book thus far. For instance, we have logical connectives: $\wedge$, $\vee$, $\sim$, $\rightarrow$, and $\leftrightarrow$, which connect sentential variables normally represented by lower case letters, e.g. p, q, and r. We also include parentheses $(,)$, (and brackets$[,]$) to clearly define the order in which to interpret the connectives and sentential variables. Furthermore, in Section 2.8, the method of truth tables was introduced to determine the truth or falsity of a compound logical sentence. We will very shortly define the formal language P, which is not precisely what we have been used to working with, but is equivalent. Remember that we have learned to rewrite specific logical connectives in terms of other logical connectives. For instance, $p \vee q \leftrightarrow \sim p \rightarrow q$ allows us to rewrite the disjunction as a conditional sentence. [1]

Definition 7.2. The *formal language* P consists of propositional symbols (lowercase letters p, q, r, etc...) along with the symbols $\sim$, $\rightarrow$, (, and), i.e. the

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logical connectives of negation and conditional, along with left and right parentheses.

As with any language, we must have clearly defined rules that tells us what is a valid, meaningful sentence. In English grammar, a complete sentence must contain at least one main clause, with a main clause containing an independent subject and verb which expresses a complete thought. For propositional languages, we have rules for how symbols can be written down in sequence. For instance, '$p \sim q$' is not a valid sentence since $\sim$ is not a binary connective (it is unary), and similarly '$p \rightarrow q)$' is also not a valid sentence because there is a missing left parenthesis. For formal language P, we can define quite readily what it means to be a well formed formula (*wff*), which is the logical equivalent of a complete sentence, grammatically speaking.

Definition 7.3. *Valid sentences* (*wffs*) in the formal language P satisfy the following criteria:
(1) A single propositional symbol of P.
(2) If A is a wff, then $\sim A$ is a wff.
(3) If A and B are wffs, then $(A \rightarrow B)$ is a wff (parentheses are sometimes dropped if $A \rightarrow B$ is alone).
(4) Nothing else is a wff of P.

Now that we have our formal language clearly defined, we are going to introduce some more terminology and symbology which will help us in exploring the concepts of completeness and consistency later on in this chapter.

Definition 7.4. An *interpretation* of a wff A of P is a row in the corresponding truth table for A.

In other words, an interpretation is specific assignment of truth values to all basic propositional symbols of A which results in a truth value for A. So an interpretation of a wff A of P will result in either 'true' or 'false'. Here, we also reiterate that we are using the standard truth tables for the connectives $\sim$ and $\rightarrow$.

Definition 7.5. A is a *logically valid formula* of P iff A is true for every interpretation in P. A logically valid formula of language P is denoted $\models_P A$.

We already have already introduced a term for the above given property, thus 'tautology' and 'logical valid formula' are equivalent concepts for our considerations. [2]

Definition 7.6. A *model-theoretically consistent* set of formulas of P is one that for which there exists at least one interpretation of P for which all formulas are true simultaneously.

Definition 7.7. A *model-theoretically inconsistent* set of formulas of P is one that for which all formulas in the set can never be simultaneously true.

Note the difference between a consistent wff and a logically valid wff. A consistent wff need only have one true value in its final truth table column, while a logically valid wff must have all true values in its final truth table column.

Definition 7.8. A formula B of P is a *semantic consequence of a formula A of* P iff there is no interpretation of P for which A is true and B is false, and is denoted $A \models_P B$.

Definition 7.9. A formula B of P is a *semantic consequence of a set of formulas Γ of* P iff there is no interpretation of P for which every formula in Γ is true and B is false. We denote this by $\Gamma \models_P B$.

7.2 Metalogical Theorems About $\models_P$

In this section we will prove theorems about logic which are fundamental to the concepts of a formal system and semantic consequence. This is different than proving theorems using logic, so pay attention to the arguments made in each of the following theorems to make sure you can follow them, in fact, we usually refer to these types of theorems as *metatheorems*, since they are theorems about theory. Many of the theorems, and subsequent proofs, will be intuitive.

Theorem P1. *If A and $A \to B$ are both true for a given interpretation, then B is true for that interpretation.*

Proof. Suppose for some interpretation A and $A \to B$ are both true, then upon inspection of the primitive truth table for $p \to q$, we see that B must also be true. □

Theorem P2. *If $\models_P A$ and $\models_P A \to B$, then $\models_P B$*

Proof. Assume that both A and $A \to B$ are logically valid while B is not. Thus, there exists an interpretation for which B is not true. For that interpretation, A will be true and $A \to B$ will be false. But this is a contradiction to our assumption that $A \to B$ was logically valid. Thus, B must be true. □

Theorem P3. $A \models_P B$ *iff* $\models_P A \to B$

Proof. We first prove that if $A \models_P B$ then $\models_P A \to B$. By assumption and the definition of semantic consequence, $A \models_P B$ implies that any time A is true, B is also true. Thus, there is no interpretation where A is true and B is false, so $A \to B$ is a logically valid formula of P, i.e. $\models_P A \to B$. Conversely, if $\models_P A \to B$, then $A \to B$ is a tautology, and thus there is no instance where A is true and B is false, therefore, $A \models_P B$. □

Theorem P4. ***The Interpolation Theorem for* P***: If $\models_P A \to B$ and A and B have at least one propositional symbol in common, then there is a formula C of* P *all of whose propositional symbols occur in both A and B such that $\models_P A \to C$ and $\models_P C \to B$.*

Proof. Suppose every propositional symbol in A also occurs in B. Then we just let C be A itself, for obviously if $\models_P A \to B$, then $\models_P A \to A$ and $\models_P A \to B$.

Now suppose that there is just one propositional symbol that occurs in A but does not occur in B. Call it p. Since by hypothesis $A \to B$ is logically valid, $A \to B$ takes the value T when p is assigned T and also takes the value T when p is assigned F. Let q be any propositional symbol that occurs in both A and B. Let A_1 be the wff that results from A when $q \to q$ is substituted for p in A, and let A_2 be the wff that results from A when $\sim (q \to q)$ is substituted for p in A. Then $A_1 \to B$ and $A_2 \to B$ are both logically valid (A_1 amounts to substituting T for p in A; A_2 to substituting F for p in A). Standard truth-table reasoning will show that, given the definitions of A_1 and A_2, $A \to (A_1 \vee A_2)$ is a truth-table tautology, and so, since $A_1 \to B$ and $A_2 \to B$ are both tautologies of P, $(A_1 \vee A_2) \to B$ is also a truth-table tautology. Now, one may cry 'foul!' since we used the disjunction operator in our argument, but $A_1 \vee A_2$ is equivalent to $\sim A_1 \to A_2$, so we can rewrite $A_1 \vee A_2$ as $\sim A_1 \to A_2$ everywhere in our argument thus far. Thus we have, if $\models_P A \to B$, then $\models_P A \to (\sim A_1 \to A_2)$ and $\models_P (\sim A_1 \to A_2) \to B$. So our C is $(\sim A_1 \to A_2)$.

If there is more than one propositional symbol that occurs in A but not in B, we let A_1 and A_2 be the formula obtained from A be replacing *each* one by $q \to q$ and $\sim (q \to q)$. The rest of the argument then goes as in the case of only one differing symbol from the previous paragraph. □

7.3 Formal Systems

Now that we have introduced the idea of a formal language, and in particular, the formal language P, we now wish to construct valid arguments using said language. To do this, we need a *deductive apparatus* – a set of axioms and rules of inference.

Definition 7.10. A *formal system* consists of a formal language and a corresponding deductive apparatus.

Referring back to Exercise 10 of Chapter 6, remember that we introduced 7 axioms. Using *instances* of axioms was a way for us to use the form of each axiom, as opposed to the specific axiom itself. For instance, Axiom I was given by $p \to (q \to p)$. All we were concerned with was the form of the axiom, not the specific sentential variable names. Thus, a specific instance of Axiom I is $\sim p \to ((q \to r) \to \sim p)$, where p is $\sim p$ and q is $q \to r$. This concept of axiomatic forms versus fixed axioms is given the name *axiomatic schema*.

We now define the formal system PS based on the formal language P. To do this, we must clearly state the axioms and the rules of inference. We will have three axiom schemas:

[PS 1] $A \to (B \to A)$
[PS 2] $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
[PS 3] $(\sim A \to \sim B) \to (B \to A)$.

The only rule of inference we will use in system PS is the Rule of Detachment (Modus Ponens). We define it precisely as follows, using the term *immediate consequence*:

Definition 7.11. (*The Rule of Detachment*) If A and B are any formulas of P, then B is an *immediate consequence* in PS of the pair of formulas A and $A \to B$.

Since we have only one rule of inference, it is all that can be used, along with axiomatic schemas, to prove anything in the formal system PS. Thus, we now are at the stage where we can introduce the definition of a proof:

Definition 7.12. A *proof in* PS is a finite string of formulas of P each one of which is an axiom of PS or an immediate consequence of two formulas preceding it in the string.

The following are examples of proofs in PS. Note that the justification for each step will be one of two possibilities, (1) an instance of one of the three axiom schemas, and (2) an application of modus ponens (an immediate consequence of two previous wffs).

Example 1.

(1)	$p \to ((p \to p) \to p)$	Axiom - PS 1
(2)	$(p \to ((p \to p) \to p)) \to$ $((p \to (p \to p)) \to (p \to p))$	Axiom - PS 2
(3)	$((p \to (p \to p)) \to (p \to p))$	MP on (1) and (2)
(4)	$(p \to (p \to p)$	Axiom - PS 1
(5)	$p \to p$	MP on (3) and (4)

Example 2.

(1)	$p \to (p \to p)$	Axiom - PS 1

Example 3.

(1)	$(\sim q \to \sim p) \to (p \to q)$	Axiom - PS 1

(2)	$((\sim q \to \sim p) \to (p \to q)) \to$ $(\sim p \to ((\sim q \to p) \to (p \to q)))$	Axiom - PS 3
(3)	$\sim p \to ((\sim q \to p) \to (p \to q))$	MP on (1) and (2)
(4)	$(\sim p \to ((\sim q \to \sim p) \to (p \to q))) \to$ $((\sim p \to (\sim q \to \sim p)) \to (\sim p \to (p \to q)))$	Axiom - PS 2
(5)	$(\sim p \to (\sim q \to \sim p)) \to (\sim p \to (p \to q)))$	MP on (3) and (4)
(6)	$\sim p \to (\sim q \to \sim p)$	Axiom - PS 1
(7)	$\sim p \to (p \to q)$	MP on (6) and (5)

The formulas mentioned in Definition 7.12 must be complete formulas, not merely subformulas of formulas. E.g. the proof in Example 1 is a string of exactly five formulas and the proof in Example 2 is a string of exactly 1 formula. By definition, a proof is a string of formulas, and the string could be written out in various ways. For instance, Example 1 could be expressed as

$$\{p \to ((p \to p) \to p), (p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p)),$$
$$((p \to (p \to p)) \to (p \to p)), (p \to (p \to p), p \to p\}$$

Definition 7.13. A formula A is a theorem of *PS* iff there is some proof in *PS* whose last formula is A. We denote this by $\vdash_{PS} A$.

From Examples 1 and 3 above, we see that $p \to p$ and $\sim p \to (p \to q)$ are theorems of *PS*. Note that we do not call each of the aforementioned wffs 'theorems', but 'theorems of *PS*'. Theorems of a formal system are dependent upon the chosen axiomatic schema and chosen rules of inference. We also might be tempted to think that $p \to (p \to p)$ is also a theorem of *PS*, and our intuition would be correct!

Definition 7.14. A string of formulas is a *derivation* in *PS* of a wff A from a set Γ of wffs of P iff:
(1) it is a finite string of formulas of P
(2) the last formula in the string is A
(3) each formula of the string is one of the following three
  (i) an axiom of *PS*
  (ii) an immediate consequence of two formulas preceding it in the string
  (iii) an element of the set Γ

Example 4. The string $\{p, p \to q, q\}$ is a derivation in *PS* of q from the set of formulas $\{p, p \to q\}$.

The difference between a derivation in *PS* and a proof in *PS* is this: In a proof in *PS*, every formula is a theorem in *PS*. In a derivation in *PS* formulas may occur in the string that are not theorems of *PS*; e.g. formulas from Γ, if

Γ is a set of formulas that are not theorems of *PS*. In the example previous, no formula in the derivation is a theorem of *PS*.

Every proof in *PS* is also a derivation in *PS* of the last formula in the proof (the theorem it proves) by setting $\Gamma = \emptyset$ (here the empty set can be denoted bt $\emptyset$ or $\{ \}$) or to any other set of wffs from P. Thus, every proof is a derivation, but not all derivations are proofs.

Definition 7.15. A formula A is a *syntactic consequence* in *PS* of a set Γ of formulas of P iff there is a derivation in *PS* of A from the set Γ. We denote this by: $\Gamma \vdash_{PS} A$

Example 5. The sentential variable r is a syntactic consequence in *PS* of the set $\Gamma = \{p, p \to q, q \to r\}$ because there is a derivation of r from that set, which we present as follows:

(1)	p	Element of Γ
(2)	$p \to q$	Element of Γ
(3)	q	MP on (1) and (2)
(4)	$q \to r$	Element of Γ
(5)	r	MP on (3) and (4)

Thus, we can express this as $\{p, p \to q, q \to r\} \vdash_{PS} r$.

A derivation is a string of formulas. A syntactic consequence is a formula that stands in a certain relation to a set of formulas.

Definition 7.16. A set Γ of formulas of P is a *proof-theoretically consistent (p-consistent)* set of *PS* iff for no formula A of P is it the case that we have both $\Gamma \vdash_{PS} A$ and $\Gamma \vdash_{PS} \sim A$.

Definition 7.17. A set Γ of formulas of P is a *proof-theoretically inconsistent (p-inconsistent)* set of *PS* iff there exists a formula A of P such that $\Gamma \vdash_{PS} A$ and $\Gamma \vdash_{PS} \sim A$.

What we call a proof-theoretically consistent set is usually called simply a consistent set. But since we already have the concept of a model-theoretically consistent set and want to distinguish the two types of consistency, we use the most natural term for doing so. It will be shown that a set of formulas is a proof-theoretically consistent set of *PS* iff it is model-theoretically consistent set of P, however the proof is rather elaborate.

A set Γ of formulas of P may be a p-inconsistent set of *PS* even though for no formula A are both A and $\sim A$ members of Γ. p-consistent and p-inconsistent sets are defined in terms of what can be derived from them with the help of the axioms and the rules of inference for *PS*, and a formula can be derived in *PS* from a set of formulas without being a member of the set.

The definition of p-consistent set of *PS* makes essential reference to the deductive apparatus of *PS*. Contrast the definition of m-consistent set, which does not.

The notion of a p-consistent set can be similarly defined for other formal systems. A set of formulas that is a p-consistent set of a formal system S may be a p-inconsistent set of another formal system S' with the same formal language as S. It depends on the deductive apparatuses of S and S'. By contrast, an m-consistent set of formulas of a formal language L remain an m-consistent set of formulas of L no matter what deductive apparatus is added to L.

7.4 Metalogical Theorems About $\vdash_{PS}$

We introduce some fairly obvious theorems about *PS* which will not include proofs, as the proofs are similar to those of Theorems P1–P4 from Section 7.2. For the following theorems, A and B are arbitrary formulas of P while Γ and Δ are arbitrary sets of formulas of P.

Theorem PS1. $A \vdash_{PS} A$

Theorem PS2. *If* $\Gamma \vdash_{PS} A$, *then* $\Gamma \cup \Delta \vdash_{PS} A$

Theorem PS3. *If* $\Gamma \vdash_{PS} A$ *and* $A \vdash_{PS} B$, *then* $\Gamma \vdash_{PS} B$

Theorem PS4. *If* $\Gamma \vdash_{PS} A$ *and* $\Gamma \vdash_{PS} A \to B$, *then* $\Gamma \vdash_{PS} B$

Theorem PS5. *If* $\vdash_{PS} A$ *then* $\Gamma \vdash_{PS} A$

Theorem PS6. $\vdash_{PS} A$ *iff* $\{\ \} \vdash_{PS} A$

Theorem PS7. $\Gamma \vdash_{PS} A$ *iff there is a finite subset* Δ *of* Γ *such that* $\Delta \vdash_{PS} A$

This metatheorem follows from the requirement that a derivation must be a finite string of formulas. There is an exact model theoretic analogue (replace $\vdash$ with $\models$ in Theorem PS7). Though: If $\Delta \vdash_{PS} A$ then $\Gamma \vdash_{PS} A$ can be proven now, the proof of the converse statement will not be given until a proof of *semantic completeness* of *PS* is presented.

In Section 6.6, we introduced the concepts of completeness and consistency. We will revisit these concepts now that we have clearly defined the meaning of a formal system.

Definition 7.18. A system S is *simply consistent* iff for no formula A of S are both A and the negation of A theorems of S.

Definition 7.19. A system S is *absolutely consistent* iff at least one formula of S is not a theorem of S.

Theorem PS8. *If* PS *is simply consistent then* PS *is absolutely consistent.*

Proof. Suppose that *PS* is simply consistent. Then for no formula A of *PS* are both A and the negation of A theorems of *PS*. So for some particular formula B of *PS*, either B is not a theorem of *PS* or $\sim B$ is not a theorem of *PS*. But both B and $\sim B$ are formulas of *PS*. So *PS* is absolutely consistent. □

Theorem PS9. PS *is simply and absolutely consistent.*

Proof. We will argue that if A is a theorem in *PS*, then $\sim A$ is not a theorem of *PS* to prove simple consistency. Once we have simple consistency, by Theorem PS8, we get absolute consistency.

Each of three axiom-schema of *PS* represents an infinite set of axioms, the A's, B's, and C's in those schemata stand for formulas that can only be either true (T) or false (F) for a fixed row in their corresponding truth table. However, since we only care about the form of our axioms, we can simply construct truth tables for PS 1, PS 2, and PS 3 in terms of sentential variables A, B, and C. Clearly these are all tautological sentences (as they should be).

We further claim that the only rule of inference (the Rule of Detachment) preserves the value of T. I.e. for any arbitrary pair of formulas A and B, if A and $A \to B$ are true, then B is also true for those rows of the truth table. This is easy to check, as we can consider the tautological sentence $(A \wedge (A \to B)) \to B$.

Using the fact that each axiom is a tautology, and that the rule of detachment preservers the truth value T, and by the definition of the process of a proof of a theorem in *PS*, any theorem must be a tautology. Furthermore, if sentence A is a theorem, by definition of the connective $\sim$ and its corresponding fundamental truth table, $\sim A$ must be false. Thus, we now know that *PS* is simply consistent, which also implies by Theorem PS8, that *PS* is absolutely consistent. [3] □

7.5 The Deduction Theorem for *PS*

The Deduction Theorem is a metatheorem that is rather useful for proving other metatheorems. It seems to have been first proved by Alfred Tarski in 1921, but the earliest published proof (for a system of predicate logic: the result includes that fora system of propositional logic) was by Jacques Herbrand in 1930. Herbrand, who made outstanding contributions to mathematical logic, was killed in a mountaineering accident in 1931, at the age of twenty-three.

Theorem PS10. ***The Deduction Theorem for*** **PS***: If* $\Gamma, A \vdash_{PS} B$*, then* $\Gamma \vdash_{PS} A \to B$

In other words: If B is a syntactic consequence in *PS* of the set $\Gamma \cup \{A\}$, then $A \to B$ is a syntactic consequence in *PS* of the set Γ alone.

This in turn means: If there is a derivation in *PS* of B from $\Gamma \cup \{A\}$, then there is a derivation in *PS* of $A \to B$ from Γ alone.

Example 6. If there is a derivation in *PS* of q from $\Gamma \cup \{p\}$, then there is a derivation in *PS* of $p \to q$ from Γ alone.

Example 7. If there is a derivation in *PS* of q from $\emptyset \cup \{p\}$, then there is a derivation in *PS* of $p \to q$ from $\emptyset$ alone.

From example 7, it can be seen that the Deductive Theorem implies that if $A \vdash_{PS} B$, then $\vdash_{PS} A \to B$

Outline of proof. The idea of the proof is this: We suppose that, for an arbitrary set Γ of formulas of P and an arbitrary formula A of P, there is a derivation in *PS* of a formula B from $\Gamma \cup \{A\}$. We can then show how, given such a derivation, to construct a derivation in *PS* of $A \to B$ from Γ alone. We conclude that if there is any derivation in *PS* of B from $\Gamma \cup \{A\}$, then there is a derivation in *PS* of $A \to B$ from Γ alone.

The proof has to show that, given *any* arbitrary derivation of B from $\Gamma \cup \{A\}$, there is a derivation of $A \to B$ from Γ alone. So what we do is, first, consider all derivations of B from $\Gamma \cup \{A\}$ that are *exactly one* formula long, and show how to construct, in any such case, a derivation of $A \to B$ from Γ (a derivation that need not be just one formula long). Then we show that if the Deduction Theorem holds for *all* derivations of B from $\Gamma \cup \{A\}$ that are less than k formulas long (where k is an arbitrary positive integer), it will hold for *all* derivations of B from $\Gamma \cup \{A\}$ that are *exactly* k formulas long. And from those two results, and the fact that, by definition, any derivation in *PS* is only finitely long, it follows that the Deduction Theorem holds for *all* derivations of B from $\Gamma \cup \{A\}$.

The form of proof described in the last paragraph is known as *proof by strong induction*. Its two stages are known as the *basis* and *induction step*. In the basis of a proof by mathematical induction we establish that the theorem under consideration holds for the minimal case (in the present theorem, the case where the derivation is one formula long: there cannot be a more minimal case than this; a derivation has to be at least one formula long). In the induction step we prove that *if* the theorem holds for all cases up to an arbitrary given point, *then* it holds at the next higher point. (This will be a *strong* induction. A *weak* induction is one in which the inductive step shows that if the theorem holds for all case at an arbitrary given point, then it holds for all cases at the next higher point.)

Proof. Let Γ be an arbitrary set of formulas of P. Let A be an arbitrary formula of P. Let D be a derivation in *PS* of a formula B from $\Gamma \cup \{A\}$. D is a finite string of formulas. If there are n formulas in D, let them be $D_1, D_2, \ldots, D_n$, in the order in which they occur in D; in such a case we shall say that D is of length n.

Basis: n=1

Let D be a derivation in *PS* of B from $\Gamma \cup \{A\}$ that is exactly one formula long. We shall show how, given D, we may construct a derivation D' (and which need not be of length 1) of $A \to B$ from Γ alone.

Since D is the derivation of B and is exactly one formula long, D must in this case consist simply of the formula B itself. That is: $D = D_n = D_1 = B$. Since there are no formulas in D preceding B, B cannot be an immediate consequence in *PS* of preceding formulas. So by the definition of derivation in *PS*, B must be either an axiom or in the set $\Gamma \cup \{A\}$. This gives us three cases to consider.

(1) B is an axiom.
(2) $B \in \Gamma$.
(3) $B = A$.

We show, in each of these cases, how to construct a derivation D' with $A \to B$ as its last formula and satisfying the condition that it is a derivation from Γ alone.

Case (1). B is an axiom. Then D' is the derivation (of length 3)

(1)	B	Axiom by assumption
(2)	$B \to (A \to B)$	Axiom - PS 1
(3)	$A \to B$	MP on (1) and (2)

Here D' is not merely a derivation but a proof in *PS*, and therefore a derivation in *PS* of its last formula from *any* set of formulas in P.

Case (2). B is in the set Γ. Then D' is

(1)	B	Element of Γ
(2)	$B \to (A \to B)$	Axiom - PS 1
(3)	$A \to B$	MP on (1) and (2)

Case (3). B is A itself. Then $A \to B$ is $A \to A$ and D' is

(1)	$A \to ((A \to A) \to A)$	Axiom - PS 1
(2)	$(A \to ((A \to A) \to A)) \to$ $((A \to (A \to A)) \to (A \to A))$	Axiom - PS 2
(3)	$(A \to (A \to A)) \to (A \to A)$	MP on (1) and (2)
(4)	$A \to (A \to A)$	Axiom - PS 1
(5)	$A \to A$	MP - (3) and (4)

This is a proof of $A \to A$ in *PS*, and therefore automatically a derivation in *PS* of $A \to A$ from any set of formulas of P you may choose.

Induction Step

Assume: The Deduction Theorem holds for every derivation of length less than k. It must be proven, on this assumption, that it holds for every derivation of length k

Let D be any derivation of B from $\Gamma \cup \{A\}$ of length K. There are four cases to consider:

(1) B is an axiom.

(2) $B \in \Gamma$.

(3) $B = A$.

(4) B is an immediate consequence by modus ponens of two proceding formulas in D.

Cases (1)–(3). D' is exactly as in the basis step.

Case (4). B is an immediate consequence by MP of two previous formulas in D. Let these two formulas be D_i and D_j where $i, j < k$ (since both D_i and D_j precede B, and B is D_k). Then either D_i is $D_j \to B$ or D_j is $D_i \to B$, since otherwise B would not be an immediate consequence by MP of D_i and D_j. It does not matter which alternative we take, so let us from now on suppose that D_j is $D_i \to B$.

The length of the derivation of D_i from $\Gamma \cup \{A\}$ is less than k. So, by assumption of the inductive step, we have that $\Gamma, A \vdash_{PS} D_i$ gives us

$$(1) \qquad \Gamma \vdash_{PS} A \to D_i$$

Using the exact same argument on D_j gives $\Gamma \vdash_{PS} A \to D_j$, but D_j is $D_i \to B$, so we have

$$(2) \qquad \Gamma \vdash_{PS} A \to (D_i \to B)$$

Using an instance of Axiom PS 2 gives

$$(3) \qquad \vdash_{PS} (A \to (D_i \to B)) \to ((A \to D_i) \to (A \to B))$$

Therefore, using Theorems PS4 and PS5 on (2) and (3) yields

$$(4) \qquad \vdash_{PS} (A \to D_i) \to (A \to B)$$

and again using Theorems PS4 and PS5 on (1) and (4)

$$(5) \qquad \vdash_{PS} (A \to B)$$

This completes the induction step, and with it the proof of the Deduction Theorem for *PS*. □

Analysis of this proof of the Deduction Theorem will show that the only special properties of *PS* that we appealed to were these:

(1) Any wff of the form $A \to (B \to A)$ is a theorem.

(2) Any wff of the form $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ is a theorem.

(3) Modus ponens for $\to$ is the *sole* rule of inference.

So our proof shows that the Deduction Theorem will hold for any other formal system having those three properties.

It is clear that if $\vdash_{PS} A \to B$ then $A \vdash_{PS} B$. From this truth and the Deduction Theorem, we get the useful metatheorem:

Theorem PS11. *$A \vdash_{PS} B$ iff $\vdash_{PS} A \to B$.*

7.6 Concepts of Semantic Completeness

Section 6.6 introduced the definition of completeness as: 'if at least one of any pair of contradictory sentences, p and $\sim p$, formulated exclusively in the terms of the theory under consideration can be proven in this theory'. Clearly from this definition, propositional logic could never be complete because there are many wffs which are neither theorems, nor their negations theorems. The cheapest of examples would be the wff p.

So what do logicians want? The Holy Grail of logic would be a system or set of systems that caught *all* the truths of pure logic. This nobody has yet found. All truths of pure propositional logic then? But they include non-truth-functional logic truths, and the task of systematising these is not yet finished (in fact has hardly yet begun). So we ask: 'What do we want that we have some hope of getting?' Here an answer is: 'All truths of pure truth-functional propositional logic'. We shall show that there is a sense in which *PS* does catch all the truths of pure truth-functional propositional logic, and we now try to make clear what this sense is.

In the first place, the language P of *PS* is adequate for the expression of any truth function. This does not mean that P is capable of expressing any truth about truth functions: it cannot, for instance, express the truth that there are a countable number of distinct truth functions. And it does not mean that every truth-functional tautology belongs to P: for P has only two connectives $\sim$ and $\to$, and so the tautology $\sim (p \wedge \sim p)$, for example, does not belong to P. But what we can get is this: Any truth-functional formula F with arbitrary connectives could be correlated with a unique formula using only $\sim$ and $\to$ connectives which has the same truth table. Nothing we say here ensures that there is a 1–1 relation between F and the formula of P: two or more distinct formulas, say F and F', might both be correlated with the same formula P. Nevertheless, to any formula F there would be correlated, in what we hope the reader will allow to be a reasonably natural way, a unique formula of P having the same truth table. So (we claim):

Every truth-functional formula is (can be) represented in a natural way by some formula of P.

Second, we shall show shortly that every tautology of P is a theorem of *PS*. Taken in conjunction with what we have said, this give us:

Every truth-functional tautology is (can be) represented in some natural way by some theorem of *PS*. Accordingly, we define the following notion of completeness for arbitrary formal systems of truth-functional propositional logic:

Definition 7.20. A formal system S with language L is *complete with respect to the class of all truth-functional tautologies* iff (1) L is adequate for the expression of any truth function and (2) every tautology of L is a theorem of S.

Similarly for *PS*:

Definition 7.21. *PS* is *complete with respect to the class of all truth-functional tautologies* iff (1) P is adequate for the expression of any truth function and (2) every tautology of P is a theorem of *PS*.

Usually, in the literature, proofs of completeness tacitly assume known results about adequacy (for the expression of any truth function) and they concentrate simply on showing that every tautology expressible in the language of the system is a theorem. We shall follow them in this, and so, taking account of the fact that for P the set of tautologies is the same as the set of all logically valid formulas, we frame the following definition:

Definition 7.22. *PS* is *semantically complete* iff every logically valid formula of P is a theorem of *PS*. (In metatheorem symbology, this is simply stated as: if $\models_P A$ then $\vdash_{PS} A$).

It should be remarked that there is no universal agreement on the definition of 'semantic completeness'. The phrase will not necessarily be understood by all workers in the field. Most workers simply write 'complete' and then give a brief explanation, or they leave it to the context to reveal what sense of 'complete' they intended.

There are several proofs of the semantic completeness of *PS*, and we will use an approach similar to that of László Kalmár who did so in 1935. For the proof, we need to know that for arbitrary wffs A, B and C of P the following are theorems of *PS*:

[Item 1] $A \rightarrow A$
[Item 2] $A \rightarrow (B \rightarrow A)$
[Item 3] $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
[Item 4] $\sim A \rightarrow (A \rightarrow B)$
[Item 5] $A \rightarrow \sim\sim A$
[Item 6] $A \rightarrow (\sim B \rightarrow \sim (A \rightarrow B))$

[Item 7] $(A \to B) \to ((\sim A \to B) \to B)$

Kalmár's approach to the proof will work for any system for which all formulas of any of those seven patterns are theorems and MP is a rule of inference. Showing that all formulas of P of any of those seven patterns are theorems is a rather painful exercise involving no new fundamental ideas, but we will reproduce the work in the following theorems. Our goal is to derive each of Items 1 through Item 7 using PS 1–3, MP, and any other *PS* related metatheorems we have proven thus far.

Item 1 we have already proven in the course of proving the Deduction Theorem of the previous section (see Case (3) of the basis step). Items 2 and 3 are just the axiom-schemata PS 1 and PS 2. That leaves Items 4–7 to be proved. It will help in proving these four metatheorems if we first prove metatheorem PS11.

The proofs of Items 4–7 are proofs in the metalanguage to the effect that certain formulas are theorems of *PS*. Throughout what follows, A, B, C are to be arbitrary formulas of P, and 'DT' is the abbreviation for the Deduction Theorem.

Theorem PS11. $\{A \to B, B \to C\} \vdash_{PS} A \to C$

Proof.

(1)	A	Assumption
(2)	$A \to B$	Assumption
(3)	B	MP on (1) and (2)
(4)	$B \to C$	Assumption
(5)	C	MP on (3) and (4)
(6)	$\{A \to B, B \to C\} \vdash_{PS} C$	Def. of $\vdash_{PS}$
(7)	$\{A \to B, B \to C\} \vdash_{PS} A \to C$	DT on (6)

□

Theorem (Item 4). $\sim A \to (A \to B)$

Proof.

(1)	$\sim A \to (\sim B \to \sim A)$	Axiom - PS 1
(2)	$(\sim B \to \sim A) \to (A \to B)$	Axiom - PS 3
(3)	$\sim A \to (A \to B)$	Thm. PS11 on (1), (2)

□

Theorem PS12. $\sim\sim A \to A$

Proof.

(1)	$\sim\sim A \to (\sim A \to \sim\sim\sim B)$	Item 4
(2)	$(\sim A \to \sim\sim\sim A) \to (\sim\sim A \to A)$	Axiom - PS 3
(3)	$\sim\sim A \to (\sim\sim A \to A)$	Thm. PS11 on (1), (2)
(4)	$(\sim\sim A \to (\sim\sim A \to A)) \to$ $((\sim\sim A \to \sim\sim A) \to (\sim\sim A \to A))$	Axiom - PS 2
(5)	$(\sim\sim A \to \sim\sim A) \to (\sim\sim A \to A)$	MP on (3) and (4)
(6)	$\sim\sim A \to \sim\sim A$	Item 1
(7)	$\sim\sim A \to A$	MP on (5) and (6)

□

Theorem (Item 5). $A \to \sim\sim A$

Proof.

(1)	$\sim\sim\sim A \to \sim A$	Thm. PS12
(2)	$(\sim\sim\sim A \to \sim A) \to (A \to \sim\sim A)$	Axiom - PS 3
(3)	$A \to \sim\sim A$	MP on (1), (2)

□

Theorem PS13. $\{\sim\sim A, A \to B\} \vdash_{PS} \sim\sim B$

Proof.

(1)	$\sim\sim A$	Assumption
(2)	$\sim\sim A \to A$	Thm. PS12
(3)	A	MP on (1) and (2)
(4)	$A \to B$	Assumption
(5)	B	MP on (3) and (4)
(6)	$B \to \sim\sim B$	Thm. (Item 5)
(7)	$\sim\sim B$	MP on (5) and (6)
(8)	$\{\sim\sim A, A \to B\} \vdash_{PS} \sim\sim B$	Def. of $\vdash_{PS}$

□

Theorem PS14. $\vdash_{PS} (A \to B) \to (\sim\sim A \to \sim\sim B)$

Proof.

(1)	$\{\sim\sim A, A \to B\} \vdash_{PS} \sim\sim B$	Thm. PS13

(2) $\{A \to B\} \vdash_{PS} \sim\sim A \to \sim\sim B$ DT on (1)
(3) $\vdash_{PS} (A \to B) \to (\sim\sim A \to \sim\sim B)$ DT on (2)

$\square$

Theorem PS15. $\vdash_{PS} (A \to B) \to (\sim B \to \sim A)$

Proof.

(1) $\vdash_{PS} (A \to B) \to (\sim\sim A \to \sim\sim B)$ Thm. PS14
(2) $\vdash_{PS} (\sim\sim A \to \sim\sim B) \to (\sim B \to \sim A)$ Axiom - PS 3
(3) $\vdash_{PS} (A \to B) \to (\sim B \to \sim A)$ Thm. PS11 on (1), (2)

$\square$

Theorem (Item 6). $\vdash_{PS} A \to (\sim B \to \sim (A \to B))$

Proof.

(1) $\{A, A \to B\} \vdash_{PS} B$ Def. of MP
(2) $A \vdash_{PS} (A \to B) \to B$ DT on (1)
(3) $\vdash_{PS} A \to ((A \to B) \to B)$ DT on (2)
(4) $\vdash_{PS} ((A \to B) \to B) \to (\sim B \to \sim (A \to B))$ Thm. PS15
(5) $\vdash_{PS} A \to (\sim B \to \sim (A \to B))$ Thm. PS11 (3), (4)

$\square$

Theorem PS16. $\vdash_{PS} (\sim A \to A) \to (B \to A)$

Proof.

(1) $\vdash_{PS} \sim A \to (B \to A)$ Item 4
(2) $\vdash_{PS} (\sim A \to (A \to \sim B)) \to$
$((\sim A \to A) \to (\sim A \to \sim B))$ Axiom - PS 2
(3) $\vdash_{PS} ((\sim A \to A) \to (\sim A \to \sim B))$ MP on (1) and (2)
(4) $\vdash_{PS} (\sim A \to \sim B) \to (B \to A)$ Axiom - PS 3
(5) $\vdash_{PS} (\sim A \to A) \to (B \to A)$ Thm. PS11 on (3), (4)

$\square$

Theorem PS17. $\vdash_{PS} (\sim A \to A) \to A$

Proof.

(1)	$\vdash_{PS} (\sim A \to A) \to ((\sim A \to A) \to A)$	Th. PS16
(2)	$\vdash_{PS} ((\sim A \to A) \to ((\sim A \to A) \to A) \to A)) \to$ $(((\sim A \to A) \to (\sim A \to A)) \to$ $((\sim A \to A) \to A))$	Axiom - PS 2
(3)	$\vdash_{PS} (((\sim A \to A) \to (\sim A \to A)) \to$ $((\sim A \to A) \to A))$	MP (1), (2)
(4)	$\vdash_{PS} (\sim A \to A) \to (\sim A \to A)$	Item 1
(5)	$\vdash_{PS} (\sim A \to A) \to A$	MP (3), (4)

□

Theorem PS18. $\vdash_{PS} (\sim B \to \sim A) \to ((\sim A \to B) \to (\sim B \to B))$

Proof.

(1)	$\sim B$	Assumption
(2)	$\sim B \to \sim A$	Assumption
(3)	$\sim A$	MP on (1), (2)
(4)	$\sim A \to B$	Assumption
(5)	B	MP on (3), (4)
(6)	$\{\sim B \to \sim A, \sim A \to B, \sim B\} \vdash_{PS} B$	Def. of $\vdash_{PS}$
(7)	$\{\sim B \to \sim A, \sim A \to B\} \vdash_{PS} \sim B \to B$	DT on (6)
(8)	$\{\sim B \to \sim A\} \vdash_{PS} (\sim A \to B) \to (\sim B \to B)$	DT on (7)
(9)	$\vdash_{PS} (\sim B \to \sim A) \to ((\sim A \to B) \to (\sim B \to B))$	DT on (8)

□

Theorem PS19. $\{A \to B, \sim A \to B\} \vdash_{PS} B$

Proof.

(1)	$A \to B$	Assumption
(2)	$(A \to B) \to (\sim B \to \sim A)$	Thm. PS15
(3)	$\sim B \to \sim A$	MP on (1), (2)
(4)	$(\sim B \to \sim A) \to ((\sim A \to B) \to (\sim B \to B))$	Thm. PS18
(5)	$(\sim A \to B) \to (\sim B \to B)$	MP on (3), (4)
(6)	$\sim A \to B$	Assumption
(7)	$\sim B \to B$	MP on (5), (6)

(8)	$(\sim B \to B) \to B$	Thm. PS17
(9)	B	MP on (7), (8)
(10)	$\{A \to B, \sim A \to B\} \vdash_{PS} B$	Def. of $\vdash_{PS}$

□

Theorem (Item 7). $\vdash_{PS} (A \to B) \to ((\sim A \to B) \to B)$

Proof.

(1)	$\{A \to B, \sim A \to B\} \vdash_{PS} B$	Thm. PS19
(2)	$\{A \to B\} \vdash_{PS} (\sim A \to B) \to B$	DT on (1)
(3)	$\vdash_{PS} (A \to B) \to ((\sim A \to B) \to B)$	DT on (2)

□

That completes the proof that any formula of the pattern of any of Items 1–7 is a theorem of *PS*. We can now go on to state and prove that lemma that is the heart of Kalmárs proof. The most difficult thing about the whole proof is understanding what it is that the lemma asserts.

Lemma for the Semantic Completeness Theorem. *Let A be any formula of* P *whose only distinct propositional symbols are $B_1, \ldots, B_k$ for $k \geq 1$. Let I be any interpretation (i.e. row of corresponding truth table) of* P. *I assigns a set of truth values to the propositional symbols of A, i.e. to $B_1, \ldots, B_k$. We define B_i^I as follows:*

If I assigns T to B_i, B_i^I is to be B_i itself.

If I assigns F to B_i, B_i^I is to be $\sim B_i$.

Similarly, let A^I be either A or $\sim A$ according as A is true or false for I. Then $\left\{B_1^I, \ldots, B_k^I\right\} \vdash A^I$

Explanation. Intuitively, what the Lemma says is this: Let A be any formula of P with k distinct propositional symbols. Write out the truth table for A in the standard way. Then for *each row* of the truth table a distinct syntactic consequence relation holds in *PS*. For *each row* the relation is this: If a propositional symbol is assigned T, then write it to the left of the $\vdash$ sign; if it is assigned F, then write its negation to the left of the $\vdash$ sign. If A is assigned T, write A to the right of the $\vdash$ sign. If A is assigned F, write $\sim A$ to the right of the $\vdash$ sign.

As an example, let A be $(p \to \sim q) \to \sim r$. It does not matter which of p, q, r we take to be B_1, B_2, B_3, so we let $B_1 = p$, $B_2 = q$, and $B_3 = r$. Then $A = (B_1 \to \sim B_2) \to \sim B_3$, and the truth table is given in Table 7.1.

Then the Lemma tells us that *eight* syntactic consequence relations hold in the case of A, viz.

B_1	B_2	B_3	A
T	T	T	T
F	T	T	F
T	F	T	F
F	F	T	F
T	T	F	T
F	T	F	T
T	F	F	T
F	F	F	T

Table 7.1: Derivative truth table for $(B_1 \to \sim B_2) \to \sim B_3$

[Row 1] $\{p, q, r\} \vdash_{PS} (p \to \sim q) \to \sim r$
[Row 2] $\{\sim p, q, r\} \vdash_{PS} \sim ((p \to \sim q) \to \sim r)$
[Row 3] $\{p, \sim q, r\} \vdash_{PS} \sim ((p \to \sim q) \to \sim r)$
[Row 4] $\{\sim p, \sim q, r\} \vdash_{PS} \sim ((p \to \sim q) \to \sim r)$
[Row 5] $\{p, q, \sim r\} \vdash_{PS} (p \to \sim q) \to \sim r$
[Row 6] $\{\sim p, q, \sim r\} \vdash_{PS} (p \to \sim q) \to \sim r$
[Row 7] $\{p, \sim q, \sim r\} \vdash_{PS} (p \to \sim q) \to \sim r$
[Row 8] $\{\sim p, \sim q, \sim r\} \vdash_{PS} (p \to \sim q) \to \sim r$

It is vital to see the following point: *Every syntactic consequence relation holds simply in virtue of the syntactic properties of* PS. In proving the Lemma we refer to interpretations of P, but every syntactic consequence relation asserted by the Lemma *holds independently of any interpretation of* P. So, e.g., the eight syntactic consequence relations of the example belong to the pure proof theory of *PS*, and in stating them all references to interpretations of P simply drop out. The presence, both in the statement of the Lemma and in its proof, of references to interpretations of P, tends to obscure this vital point.

Proof. The proof is by induction on the number, m, of connectives in the formula A.

Basis: $m = 0$

Then A is a single unnegated propositional symbol, B_1. So the lemma reduces to $B_1 \vdash_{PS} B_1$ for the case where I assigns T to B_1 (i.e. for the row of the truth table for which B_1 gets T) and $\sim B_1 \vdash_{PS} \sim B_1$ for the case where I assigns F to B_1. Both of these hold by Theorem PS1. This completes the basis step.

Inductive Step

Assume that the Lemma holds for all formulas with fewer than m connectives (this is the induction hypothesis). To prove it holds for any arbitrary formula A with m connectives there are two cases to be considered.

1. For some formula C, A is $\sim C$, where C has fewer than m connectives.

2. For some formulas C and D, A is $C \to D$, where each of C and D has fewer than m connectives.

Case 1: For A is $\sim C$, where C has fewer than m connectives, there are two subcases.

1a. A is true for I

1b. A is false for I

Subcase 1a: A is true for I, so A^I is A. Then C is false for I, so $C^I = \sim C$. Since C has fewer than m connectives, we have, by the induction hypothesis we get $\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C^I$, but $C^I = \sim C$, so $\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim C$, but $A^I = A = \sim C$, so $\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A$. Which is what we want.

Subcase 1b: A is false for I, so A^I is $\sim A$. Then C is true for I, so $C^I = C$.

There are a few extra steps this time around, so we will write out the derivation in our standard method.

(1)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C^I$	Induction Hypothesis
(2)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C$	$C^I = C$
(3)	$C \vdash_{PS} \sim\sim C \to B)$	Item 5
(4)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim\sim C$	MP on (2), (3)
(5)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim A$	$A = \sim C$
(6)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^I$	$A = A^I$

Case 2: A is $C \to D$, where each of C and D has fewer than m connectives. For this case, there are three subcases.

2a. C is false for I

2b. D is true for I

2c. C is true for I and D is false for I

Subcase 2a: C is false for I, then A is true for I and $A^I = A = C \to D$.

(1)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C^I$	Induction Hypothesis
(2)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim C$	$C^I = \sim C$
(3)	$\vdash_{PS} \sim C \to (C \to D)$	Item 4
(4)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim C \to D$	MP on (2), (3)
(5)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim A$	$A = C \to D$
(6)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^I$	$A = A^I$

Subcase 2b: D is true for I , then A is true for I and $A^I = A = C \to D$.

(1)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} D$	Induction Hypothesis
(2)	$\vdash_{PS} D \to (C \to D)$	Axiom - PS 1
(3)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C \to D$	MP on (1), (2)

(4)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim A$	$A = C \to D$
(5)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^I$	$A = A^I$

Subcase 2c: C is true for I and D is false for I, then A is false for I and $A^I = \sim A = \sim (C \to D)$.

(1)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} C$	Induction Hypothesis
(2)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim D$	Induction Hypothesis
(3)	$\vdash_{PS} C \to (\sim D \to \sim (C \to D))$	Item 6
(4)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim D \to \sim (C \to D)$	MP on (1), (3)
(5)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim (C \to D)$	MP on (2), (4)
(6)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} \sim A$	$A = \sim (C \to D)$
(7)	$\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^I$	$A = A^I$

This completes the induction step, and with it, the proof of the Lemma. □

Theorem PS20. ***The Semantic Completeness Theorem for* PS**: *Every logically valid formula of* P *is a theorem of* PS. *In metalogic symbolism: If* $\models_P A$ *then* $\vdash_{PS} A$.

Proof. Let A be an arbitrary logically valid formula of P with distinct propositional symbols $B_1, \ldots, B_k$, with $k \geq 1$.

Let I and J be interpretations of P that differ only in that I assigns T to B_k while J assigns F to B_k. Then by the Lemma

(1) $\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^I$

(2) $\{B_1^I, \ldots, B_k^I\} \vdash_{PS} A^J$.

Now we have from our definition of I, J, etc...

(3) $B_1^J = B_1^I, \ldots, B_{k-1}^J = B_{k-1}^I$

and

(4) $B_K^I = B$

and

(5) $B_K^J = \sim B$.

Also, since A is logically valid it is true for I and true for J. So

(6) $A^I = A^J = A$.

From (1) and (4) and (6) we get

(7) $\{B_1^I, \ldots, B_{k-1}^I, B_k\} \vdash_{PS} A$.

From (2) and (3) and (5) and (6) we get

(8) $\{B_1^I, \ldots, B_{k-1}^I, \sim B_k\} \vdash_{PS} A$.

By applying the Deduction Theorem to (7) and (8) we get

(9) $\{B_1^I, \ldots, B_{k-1}^I\} \vdash_{PS} B_k \to A$

and

(10) $\{B_1^I, \ldots, B_{k-1}^I\} \vdash_{PS} \sim B_k \to A$.

But by Item 7, we have

(11) $\vdash_{PS} (B_k \to A) \to ((\sim B_k \to A) \to A)$.
So by two applications of MP, we get
(12) $\{B_1^I, \ldots, B_{k-1}^I\} \vdash_{PS} A$.

We have now eliminated B_k from the argument, If $k = 1$, we have immediately $\vdash_{PS} A$. If $k > 1$, the let L be an interpretation that differs from I only in the truth value it assigns B_{k-1} (i.e. if I assigns T to B_{k-1}, then L assigns F to B_{k-1}; and if I assigns F to B_{k-1}, then L assigns T to B_{k-1}). By repeating the set of moves made above, with obvious changes, we can eliminate B_{k-1} from the argument just as we eliminated B_k. And so on, until we have eliminated everything to the left of the $\vdash_{PS}$ sign and are left simply with $\vdash_{PS} A$. But A was an arbitrary logically valid formula of P. So if $\models_P A$ then $\vdash_{PS} A$.

□

Analysis of this proof of completeness shows that any formal system with the language P that has Modus Ponens as a rule of inference and satisfies Items 1–7 will be semantically complete.

Exercises

1. Determine whether each of the following expressions are wffs of P. If an expression is not a valid wff of P, explain why.

(a) $\sim p \to (q \to \sim r)$ (b) $\sim (p \to (q \to \sim r))$ (c) $\sim p \vee q$

(d) $p \to q \to r$ (e) $\sim p \to \sim q$ (f) $A \to B$

(g) $(p \to q) \to (p \to q)$ (h) $(\sim p) \to q$ (i) $(\sim p) \to (q \vee r)$

2. Define the proposition language we have been using thus far (previous to this chapter) clearly, along with what constitutes a wff in this propositional language.

3. For each of the following sets of formulas, determine if they are model theoretically consistent or inconsistent.

(a) $\{p \to (\sim p \to q), p \to (\sim q \to p), p\}$
(b) $\{p \to q, r \to \sim q, (p \to \sim r) \to (q \to r), (r \to \sim q) \to q\}$
(c) $\{\sim p \to q, \sim q \to r, p \to (q \to r), (\sim r \to q) \to p\}$
(d) $\{(p \to \sim q) \to r, \sim r \to (p \to q), (p \to \sim r) \to (q \to r)\}$

4. Argue that the following metalogical statements are true. Here A and B are arbitrary formulas of P while Γ and Δ are arbitrary sets of formulas of P.

(a) $A \models_P A$
(b) If $\Gamma \models_P A$, then $\Gamma \cup \Delta \models_P A$
(c) If $\Gamma \models_P A$ and $A \models_P B$, then $\Gamma \models_P B$
(d) If $\Gamma \models_P A$ and $\Gamma \models_P A \to B$, then $\Gamma \models_P B$

(e) If $\models_P A$ then $\Gamma \models_P A$

5. For each of the following wffs of the form $A \to B$, (1) first verify that $\models_P A \to B$, then (2) find a wff C with satisfies the conclusion of the Interpolation Theorem, i.e. a C such that $\models_P A \to C$ and $\models_P C \to B$.
(a) $(p \wedge q) \to (p \vee r)$
(b) $\sim [p \to \{(\sim q \to \sim q) \to \sim r\}] \to [\sim p \to \sim (q \to \sim s)]$
(c) $[(p \to \sim q) \to (q \to \sim p)] \to [(p \to r) \to \{(r \to s) \to (p \to s)\}]$

6. Using the Lemma for the Semantic Completeness Theorem (refer to Table 7.1), construct syntactic consequence relations corresponding to each interpretation of the following wffs. Please be sure to provide the corresponding truth table for each wff.
(a) $(\sim p \to q) \to \sim (p \to \sim q)$
(b) $(p \to (q \to \sim p)) \to r$

Notes

[1]The chapter is based off of Geoffrey Hunter's excellent book *Metalogic: An Introduction to the Metatheory of Standard First Order Propositional Logic*, which spends much of its time delving into the deep concepts of consistency, completeness, and decideability.

[2]It may seem like there are no other ways to view the concepts of 'tautology' and 'logically valid formula', and that we are unnecessarily making distinctions. However, we must remember that there is more than propositional logic. In Part III, we will introduce a theory which involves quantified statements, in which case we can no longer argue via the method of truth tables, but rather by interpretation.

[3]Emil Post (1920) was the first to give a proof of the consistency of standard propositional calculus. His proof was semantic in character: he showed that all theorems of the system he was dealing with (the propositional systemof *Principia Mathematica*) are tautologies. Jan Łukasiewicz seems to have been the first to find a purely syntactic proof. The key idea common to both of these consistency proofs, i.e. the idea of showing that there is some property that belongs to every axiom, is preserved by rules of inference, and does not belong to any contradictory formula) goes back to David Hilbert in 1904.

Part III

Applications of Logic and Methodology: Constructing Mathematical Theories

Chapter 8

Construction of a Mathematical Theory: Laws of Order for Numbers

8.1 Primitive Terms and Axioms on Fundamental Relations Among Numbers

With a certain amount of knowledge of the fields of logic and methodology at our disposal, we shall now undertake to lay the foundations of a particular and, incidentally, very elementary mathematical theory. This will be a good opportunity for us to assimilate better our previously acquired knowledge, and even to expand it to some extent.

The theory with which we shall concern ourselves constitutes a fragment of the arithmetic of real numbers. It contains fundamental theorems concerning the basic relations *less than* and *greater than* among numbers, namely of addition and subtraction. It presupposes nothing but logic.

The primitive terms which we shall adopt in this theory are the following:

real number,
is less than,
is greater than,
sum.

Instead of "*real number*" we shall, as before, simply say "*number*". Also, it is slightly more convenient to consider, instead of the term "*number*", the expression "*the set of all numbers*" as a primitive term, which, for brevity, we

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will replace by the symbol $\mathbb{R}$; thus, in order to express that x is a (real) number, we write:

$$x \in \mathbb{R}.$$

We may, on the other hand, stipulate that the universe of discourse of our theory consists of real numbers only and that variables such as "x", "y", ... stand exclusively for the name of numbers; in this case, the term "*real number*" would be altogether dispensable in the formulation of statements of our theory, and the symbol "$\mathbb{R}$" might, when needed, be replaced by "U" (cf. Section 4.3).

The expressions "*is less than*" and "*is greater than*" are to be treated as if they were entities consisting of a single word each; they will be replaced by the briefer symbols "$<$" and "$>$", respectively. Instead of "*is not less than*" and "*is not greater than*" we shall employ the usual symbols "$\not<$" and "$\not>$". Further, instead of "*the sum of the numbers x and y*" we shall use the customary notation

$$x + y.$$

Thus, the symbol $\mathbb{R}$ designates a certain set, the symbols "$<$" and "$>$" certain two-termed relations, and finally the symbol "$+$" a certain binary operation.

Among the axioms of the theory under consideration, two groups may be distinguished. The Axioms of the first group express fundamental properties of the relations "$<$" and "$>$", whereas those of the second are primarily concerned with addition. For the time being we shall consider the first group only; it consists, altogether, of five statements:

Axiom 1. $\forall\, x, y\ (x = y \vee x < y \vee x > y)$

Axiom 2. $x < y \to y \not< x$

Axiom 3. $x > y \to y \not> x$

Axiom 4. $(x < y \wedge y < z) \to x < z$

Axiom 5. $(x > y \wedge y > z) \to x > z$

The axioms listed here, just as any arithmetical theorem of a universal character stating that arbitrary numbers x, y, ... have such and such a property, should really begin with $\forall\, x, y, \ldots$ or $\forall\, x, y, \ldots \in \mathbb{R}$. But since we want to conform to the usage in Section 1.3, we often omit a phrase and merely add it in our mind; this holds not only for the axioms but also for the theorems and definitions which will occur in the course of our consideration. Axiom 2, is meant to be read as follows:

For any x and y of $\mathbb{R}$, if $x < y$ then $y \not< x$.

We shall refer to Axiom 1 as the *Weak Law of Trichotomy* (with the *Strong Law of Trichotomy* we shall become acquainted with later). Axioms 2-5 express that the relations $<$ and $>$ are asymmetrical and transitive (cf. Section 5.3); accordingly they are called the *Laws of Asymmetry* and *Laws of Transitivity* (for the relations $<$ and $>$). The axioms of the first group and the theorems following from them are called the *Laws of Order for Numbers*.

The relations $<$ and $>$, together with the logical relation of identity $=$, will be here referred to as the *Fundamental Relations Among Numbers.*

8.2 Laws of Irreflexivity and Indirect Proofs

Our next task in the derivation of a number of theorems from the axioms adopted by us. Since we do not aim at a systematic presentation, in this and the following chapter only those theorems will be stated which may serve to help illustrate certain concepts and facts of the fields of logic and methodology. Before we do this, however, we introduce an important logical law whose structure will come in handy when attempting to prove the first theorem based on this axiomatic system.

Law 8.1 (Law of Reductio ad Absurdum). [1] $[p \to (\sim p)] \to (\sim p)$

We shall next prove the following theorem:

Theorem 1. $x \nless x$

Proof. We first perform the proof in a general manner, and then transform this argument into a complete proof, using for clarity the logical symbolism introduced in Sections 2.8 and 2.10. To start, we will suppose that our theorem is false, i.e. that there would be a number x satisfying the formula:

$$x < x \tag{1}$$

Now Axiom 2 refers to arbitrary numbers x and y (which need not be distinct), so that it remains valid if in place of "y" we write the variable "x"; we then obtain:

$$x < x \to x \nless x. \tag{2}$$

But from 1 and 2 and the Rule of Detachment, it follows immediately that:

$$x \nless x;$$

this consequence, however, forms an obvious contradiction to formula 1. We must, therefore, reject the original assumption and accept the theorem as proved.

We shall now transform this argument into its symbolic logic form:

(1) $x < x \to x \nless x$ Instance of Axiom 2, $y : x$

(2) $[(x < x) \to (x \nless x)] \to (x \nless x)$ Law of Reductio ad Absurdum, $p : (x < x)$

(3) $x \nless x$ Rule of Detachment on (1) and (2)

□

The proof of Theorem 1 represents an example of what is called an *indirect proof*. There are many ways to perform an indirect proof, and here we used the Law of Reductio ad Absurdum. Indirect proofs are very common in mathematics. They do not all fall under the schema of the proof of Theorem 1, however; on the contrary, the latter represents a comparatively rare form of an indirect proof. In general, indirect proofs of this kind may quite generally be characterized as follows: in order to prove a theorem, we assume the theorem to be false, and derive from that certain consequences which compel us to reject the original assumption. We shall meet with more typical indirect proofs further below.

The axiomatic system adopted by us is perfectly symmetrical with respect to the symbols "$<$" and "$>$". Thus, to every theorem concerning the relation *less than*, we therefore automatically obtain the corresponding theorem concerning the relation *greater than*, the proofs being entirely analogous, so that the proof of the second theorem may be omitted altogether. In particular, corresponding to 1 we have:

Theorem 2. $x \ngtr x$

While the relation of identity $=$, as we know from logic, is reflexive, Theorems 1 and 2 show that the other two fundamental relations among numbers, $<$ and $>$, are irreflexive; these theorems are therefore called the *Laws of Irreflexivity* (for the relations *less than* and *greater than*).

8.3 Further Theorems on the Fundamental Relations

We shall next prove the following theorem:

Theorem 3. $x > y \leftrightarrow y < x$

Before we get to the proof of this theorem, we introduce a new technique to our collection of proof completing skills, which is also in the family of indirect proofs. The idea is as follows: suppose we wish to prove that a certain property holds based on a given assumption, while at the same time, there are only a finite number of possible properties to begin with, such as in the case of Axiom 1 in regards to relations among numbers. Instead of directly proving the desired property, we can *disprove* all of the other remaining properties by assuming them to be true and deriving a contradiction. These secondary assumptions, once made, must yield a contradiction, but the contradiction itself may not necessarily be immediately related to the secondary assumption. It may also be the case that more than one secondary assumption must be made in order to complete the proof, but once again, all secondary assumptions must arrive at a contradiction.

To begin the proof of Theorem 3, we must first rewrite the biconditional statement as the conjunction of two conditional sentences:

$$(x > y \to y < x) \land (y < x \to x > y)$$

To prove the theorem, we simply prove each of the conditional statements given. We start with the second statement, and assume its hypothesis. Furthermore, we will also make use of the following pair of identities related to the disjunction of three expressions (for manipulation of Axiom 1):

$$\begin{aligned} p \lor q \lor r &\leftrightarrow \sim p \to q \lor r \\ p \lor q \lor r &\leftrightarrow (\sim p \land \sim q) \to r \end{aligned} \tag{8.1}$$

Proof.

(1)	$y < x$	Assume hypothesis
(2)	$y < x \to x \not< y$	Instance of Axiom 2, $y : x, x : y$
(3)	$x \not< y$	Rule of Detachment, (1) and (2)
(4)	$x \not< y \to (x = y \lor x > y)$	Axiom 1 via 8.1
(5)	$x = y \lor x > y$	Rule of Detachment, (3) and (4)
(6)	$x = y \lor x > y \leftrightarrow x \not> y \to x = y$	Instance of $p \lor q \leftrightarrow \sim p \to q$ $p : x > y, q : x = y$
(7)	$x \not> y \to x = y$	Substitute (6) into (5)
(8)	$x \not> y$	Secondary Assumption
(9)	$x = y$	Rule of Detachment, (7) and (8)
(10)	$y < y$	Rule of Replacement, (9) and (1)
(11)	$x = y \to y < y$	Instance of $\text{True}_1 \to \text{True}_2$ $\text{True}_1 : x = y$, $\text{True}_2 : y < y$
(12)	$y \not< y$	Instance of Theorem 1, $x : y$
(13)	$y \not< y \to x \neq y$	Law of Contraposition on (11)

(14) $x \neq y$ Rule of Detachment, (12) and (13)

(15) $x \neq y \to x > y$ Law of Contraposition on (7)

(16) $x > y$ Rule of Detachment, (14) and (15)

Note that we have finally arrived at $x > y$ after exhausting the other two possibilities, namely: $x < y$ and $x = y$. We were able to deduce $x \not< y$ directly from assuming the hypothesis and Axiom 2. To arrive at $x \neq y$, we had to make the secondary assumption that $x \not> y$ to get $x = y$ and then derive a contradiction via an instance of Theorem 1. We finally arrive at $x > y$.

Of course, this is only the first half of the proof, we must still argue that $x > y \to y < x$. This however, is done exactly the same way as the first half, due to symmetry, and is left as a quick exercise for the interested reader. □

Using the terminology of the calculus of relations (cf. Section 5.2), we may say that, according to Theorem 3, each of the relations $>$ and $<$ is the converse of the other.

Theorem 4. $x \neq y \to (x < y \vee y < x)$

Proof.

(1) $x \neq y$ Assume hypothesis

(2) $x \neq y \to (x < y \vee x > y)$ Axiom 1 via 8.1

(3) $x < y \vee x > y$ Rule of Detachment, (1) and (2)

(4) $x > y \leftrightarrow y < x$ Theorem 3

(5) $x < y \vee y < x$ Substitute (4) into (3)

□

Analogously we can prove (but won't):

Theorem 5. $x \neq y \longleftrightarrow (x > y \vee y > x)$

By Theorems 4 and 5 the relations $<$ and $>$ are connected; accordingly these theorems are known as the *Laws of Connectivity* (for the relations *less than* and *greater than*). Axioms 2-5, together with Theorems 4 and 5, show that the set of real number $\mathbb{R}$ is ordered by either of the relations $<$ and $>$.

Theorem 6. *For any numbers $x, y \in \mathbb{R}$, exactly one of the three formulas: $x = y$, $x < y$ and $x > y$ is satisfied.*

Proof. To prove this theorem, we must prove the following three conditional sentences:

$$\text{(a) } x = y \rightarrow x \not< y, \quad \text{(b) } x = y \rightarrow x \not> y, \quad \text{(c) } x < y \rightarrow x \not> y \tag{8.2}$$

To see why these are the only three we need to prove, we consider the three equivalent contrapositive statements:

$$\text{(a}'\text{) } x < y \rightarrow x \neq y, \quad \text{(b}'\text{) } x > y \rightarrow x \neq y, \quad \text{(c}'\text{) } x > y \rightarrow x \not< y \tag{8.3}$$

Of course, if we prove (a), then (b) is done similarly. So what we need to do is prove (a) and (c), and the proof for (a) uses an approach similar to that done already in this chapter. So we start with the proof of (a):

(1)	$x = y$	Assume hypothesis
(2)	$x < y$	Secondary Assumption
(3)	$x < x$	Substitute (1) into (2)
(4)	$x \not< x$	Theorem 1
(5)	$x < y \rightarrow x < x$	Instance of $\text{True}_1 \rightarrow \text{True}_2$ $\text{True}_1 : x < y$, $\text{True}_2 : x < x$
(6)	$x \not< x \rightarrow x \not< y$	Law of Contraposition on (5)
(7)	$x \not< y$	Rule of Detachment, (4) and (6)

Note that we have arrived at a contradiction to our secondary assumption (in fact we wished to prove the negation of our secondary assumption). As already stated, the proof of (b) is similar. So now we prove part (c):

(1)	$x < y$	Assume hypothesis
(2)	$y > x \leftrightarrow x < y$	Instance of Theorem 2, $x : y$, $y : x$
(3)	$y > x$	Substitute (2) into (1)
(4)	$y > x \rightarrow x \not> y$	Instance of Axiom 3, $x : y$, $y : x$
(5)	$x \not> y$	Rule of Detachment, (3) and (4)

□

Theorem 6 is called the *Strong Law of Trichotomy*, or simply, the *Law of Trichotomy*; according to this law, one and only one of the three fundamental relations holds between any two given numbers. Using the phrase "*either* ... *or* ... " in the meaning proposed in Section 2.2, we can formulate Theorem 6 in a more concise manner:

Theorem 6′**.** *For any numbers $x, y \in \mathbb{R}$, we have either $x = y$ or $x < y$ or $x > y$.*

8.4 Other Relations Among Numbers

Apart from the fundamental relations, three other relations play an important part in arithmetic. One of these is the logical relation of diversity $\neq$ which we know already; the other are the relations $\leq$ and $\geq$ which will be discussed now.

The meaning of the symbol "$\leq$" is explained by the following definition:

Definition 1. $(x \leq y) \overset{def}{\longleftrightarrow} [\,(x = y) \vee (x < y)\,]$

The formula $x \leq y$ is to be read: "*x is less than or equal to y*" or "*x is at most equal to y*".

Although the content of the definition as stated appears to be clear, experience shows that in practical applications it sometimes becomes the source of certain misunderstandings. Some people who believe they understand the meaning of the symbol "$\leq$" perfectly will protest, nevertheless, against its application to definite numbers. They do not only reject a formula like:

$$1 \leq 0$$

as obviously false – and this rightly so –, but they also consider as meaningless or even false such formulas as:

$$0 \leq 0 \text{ or } 0 \leq 1;$$

for they maintain that there is no sense in saying that $0 \leq 0$ or that $0 \leq 1$ since it is known that $0 = 0$ and $0 < 1$. In other words, it is not possible to exhibit a single pair of numbers which, in their opinion, satisfies the formula $x \leq y$.

This view is palpably mistaken. Just because $0 < 1$ holds, it follows that the sentence:

$$0 = 1 \text{ or } 0 < 1$$

is true, for the disjunction of two sentences is certainly true provided one of them is true (cf. Section 2.2); but according to Definition 1 this disjunction is equivalent to the formula:

$$0 \leq 1.$$

For a quite analogous reason the formula:

$$0 \leq 0$$

is also true.

The source of these misunderstandings, presumably, lies in certain habits of everyday life. In ordinary language it is customary to assert the disjunction of two sentences only if we know that one of the sentences is true without knowing which. It does not occur to us to say that $0 = 1$ *or* $0 < 1$, though this is undoubtedly true, since we can say something that is simpler and at the same time logically stronger, namely, that $0 < 1$. In mathematical considerations, however, it is not always advantageous to state everything that we know in its strongest possible form. For example, we sometimes assert of a quadrilateral merely that it is a parallelogram, although we know it to be a square, and this because we may want to apply a general theorem concerning arbitrary parallelograms. For similar reasons it may occur that it is known of a number x (for instance, of the number 0) that it is less than 1, and yet it may merely be asserted that $x \leq 1$, that is, that either $x = 1$ or $x < 1$.

We will now state two theorems concerning the relation $\leq$.

Theorem 7. $x \leq y \leftrightarrow x \not> y$

Proof. Once again, with the biconditional theorem, we have to prove two conditional statements:

$$(x \leq y \rightarrow x \not> y) \wedge (x \not> y \rightarrow x \leq y)$$

The proof of the second half of the conjunction is rather straightforward due to Axiom 1:

(1)	$x \not> y$	Assume hypothesis
(2)	$x \not> y \rightarrow x < y \vee x = y$	Axiom 1
(3)	$x < y \vee x = y$	Rule of Detachment, (1) and (2)
(4)	$x < y \vee x = y \leftrightarrow x \leq y$	Definition 1
(5)	$x \leq y$	Substitute (4) into (3)

To prove the first conditional sentence: $x \leq y \rightarrow x \not> y$, we simply apply the Strong Law of Trichotomy (Theorem 6) to the contrapositive statement:

$$x > y \rightarrow x \not\leq y.$$

(1)	$x > y$	Assume hypothesis

(2)	$x > y \to (x \not< y \wedge x \neq y)$	Theorem 6
(3)	$x \not< y \wedge x \neq y$	Rule of Detachment, (1) and (2)
(4)	$(x \not< y \wedge x \neq y) \leftrightarrow \sim (x < y \vee x = y)$	Instance of $\sim p \wedge \sim q \leftrightarrow \sim (p \vee q)$ $p : x < y,\ q : x = y$
(5)	$\sim (x < y \vee x = y)$	Substitute (4) into (3)
(6)	$x < y \vee x = y \leftrightarrow x \leq y$	Definition 1
(7)	$\sim x \leq y$	Substitute (6) into (5)

Of course, $\sim x \leq y$ is by symbolic definition $x \nleq y$, thus (7) is indeed the end of the proof. Note that the form of the Strong Law of Trichotomy was just one of several possible ways to express it. It was in our interest to state it in the form of (2): $x > y \to (x \not< y \wedge x \neq y)$, since if $x > y$, the other two relations cannot hold. □

In the terminology of Section 5.2, Theorem 7 states that the relation $\leq$ is the negation of the relation $>$.

On account of its structure, Theorem 7 might be looked upon as the definition of the symbol "$\leq$"; it would be a different one from that adopted here but equivalent to it. The statement of the theorem may also contribute to dispel any last doubts about the usage of the symbol "$\leq$"; for nobody will hesitate any longer to recognize as true such formulas as:

$$0 \leq 0 \text{ and } 0 \leq 1$$

in view of the fact that they are equivalent to the formulas:

$$0 \not> 0 \text{ and } 0 \not> 1.$$

If we wished, we could avoid the use of the symbol "$\leq$" completely, by always employing "$\not>$" instead.

Theorem 8. $x < y \leftrightarrow (x \leq y \wedge x \neq y)$

Proof. As is the common approach, we have to prove two conditional statements:

$$(\, x < y \to (x \leq y \wedge x \neq y)\,) \wedge (\,(\, x \leq y \wedge x \neq y\,) \to x < y\,)$$

We start with the proof of the first conditional:

(1)	$x < y$	Assume hypothesis

(2)	$x < y \rightarrow x \neq y$	Theorem 6
(3)	$x \neq y$	Rule of Detachment, (1) and (2)
(4)	$x < y \rightarrow (x < y \vee x = y)$	Instance of $p \rightarrow (p \vee q)$ $p : x < y$, $q : x = y$
(5)	$x < y \vee x = y$	Rule of Detachment, (4) and (3)
(6)	$x < y \vee x = y \leftrightarrow x \leq y$	Definition 1
(7)	$x \leq y$	Substitute (6) into (5)
(8)	$x \leq y \wedge x \neq y$	Rule of And, Joining Together on (7) and (3)

In the above proof of the first half, note the instance of the Strong Law of Trichotomy used in (2). We now prove the second conditional sentence. This proof does not require any of the axioms of this chapter, just the definition of $\leq$ and logical manipulations.

(1)	$x \leq y \wedge x \neq y$	Assume hypothesis
(2)	$(x < y \vee x = y) \leftrightarrow x \leq y$	Definition 1
(3)	$(x < y \vee x = y) \wedge x \neq y$	Substitute (2) into (1)
(4)	$[\,(x < y \vee x = y) \wedge x \neq y\,] \leftrightarrow$ $[\,(x < y \wedge x \neq y) \vee (x = y \wedge x \neq y)\,]$	Inst. of Law of Associative And (Law 2.13) $p : x \neq y$, $q : x < y$
(5)	$(x < y \wedge x \neq y) \vee (x = y \wedge x \neq y)$	Substitute (4) into (3)
(6)	$(x = y \wedge x \neq y) \leftrightarrow \mathrm{F}$	Instance of $(p \wedge \sim p) \leftrightarrow \mathrm{F}$ $p : x = y$
(7)	$(x < y \wedge x \neq y) \vee \mathrm{F}$	Substitute (6) into (5)
(8)	$[\,(x < y \wedge x \neq y) \vee \mathrm{F}\,] \leftrightarrow$ $(x < y \wedge x \neq y)$	Instance of $(p \vee \mathrm{F}) \leftrightarrow p$ $p : (x < y \wedge x \neq y)$
(9)	$x < y \wedge x \neq y$	Substitute (8) into (7)
(10)	$(x < y \wedge x \neq y) \rightarrow x < y$	Inst. of Law of And (Law 2.1) $p : x < y$, $q : x \neq y$
(11)	$x < y$	Rule of Detachment on (9), (10)

□

Note that in sentence (5), we arrived at a compound sentence of the form $q \wedge \sim q$ on the right side of the disjunction. This is clearly false. Thus, we make use of the fact that in a disjunction, at least one sentence must be true to make the compound sentence true. Thus $p \vee F$ is equivalent to p, which gives us, after a few logical maneuvers, sentence (9).

A number of other theorems concerning the relation $\leq$ we shall pass over; among them, there are, in particular, theorems to the effect that this relation is reflexive and transitive. The proofs of none of these theorems afford any difficulties.

The definition of the symbol "$\geq$" is entirely analogous to Definition 1; and from the theorems concerning the relation $\leq$ we automatically obtain corresponding theorems concerning the relation $\geq$ by merely replacing the symbols "$\leq$", "$<$", and "$>$" throughout by the symbols "$\geq$", "$>$", and "$<$".

Formulas of the form:

$$x = y$$

in which the places of "x" and "y" may be taken by constants, variables, or compound expressions denoting numbers are usually called *equations*. Similar formulas of the form

$$x < y \text{ or } x > y$$

are called *inequalities* (*in the narrower sense*); among the *inequalities in the wider sense* we have, in addition, formulas of the form:

$$x \neq y, \quad x \leq y, \quad x \geq y.$$

The expressions occurring on the left and right sides of the symbols "$=$", "$<$", and so on, in these formulas are referred to as the *left and right sides of the equation* or *of the inequality.*

We conclude this chapter with a list of all 5 axioms and 8 theorems based on these axioms from this chapter. Having the list of all axioms and theorems all in one location will make referring and using them much easier in the following exercises.

Axiom 1. $\forall x, y \ (x = y \vee x < y \vee x > y)$

Axiom 2. $x < y \rightarrow y \nless x$

Axiom 3. $x > y \rightarrow y \ngtr x$

Axiom 4. $(x < y \wedge y < z) \rightarrow x < z$

Axiom 5. $(x > y \wedge y > z) \rightarrow x > z$

Theorem 1. $x \nless x$

Theorem 2. $x \ngtr x$

Theorem 3. $x > y \longleftrightarrow y < x$

Theorem 4. $x \neq y \longleftrightarrow (x < y \vee y < x)$

Theorem 5. $x \neq y \longleftrightarrow (x > y \vee y > x)$

Theorem 6. *For any numbers $x, y \in \mathbb{R}$, exactly one of the three formulas: $x = y$, $x < y$ and $x > y$ is satisfied.*

Theorem 7. $x \leq y \leftrightarrow x \not> y$

Theorem 8. $x < y \leftrightarrow (x \leq y \wedge x \neq y)$

Exercises

1. Consider two relations among people: that of being of a smaller stature, and that of being of a larger stature. What condition has to be satisfied by an arbitrary set of people, so that it together with those two relations, forms a model of the first group of axioms (cf. Section 6.2)?

2. From Theorem 1 derive the following theorem:

Theorem A. $x < y \rightarrow x \neq y$

3. Conversely, derive Theorem 1 from the theorem just proved in Exercise 3 without making use of any other arithmetical statements.

4. Show that, if Theorem 1 is adopted as a new axiom, the old Axiom 2 can be derived as a theorem from this axiom together with Axiom 4.

5. Derive the following theorems from the first group of axioms:

Theorem B. $x = y \leftrightarrow (x \not< y \wedge y \not< x)$

Theorem C. $x < y \rightarrow (x < z \vee z < y)$

6. Show that, between any two numbers, exactly three of the following six relations hold: $=$, $<$, $>$, $\neq$, $\leq$, and $\geq$.

7. Both the converse and the negation of any of the relations listed in the preceding exercise are again among these six relations. Show in detail that this is the case.

Notes

[1]This law, together with a related one of the same name:

$$(\sim p \rightarrow p) \rightarrow p,$$

has been used in many intricate and historically important arguments in logic and mathematics. The Italian logician Vailati (1863-1909) devoted a special monograph to its history.

Chapter 9

Construction of a Mathematical Theory: Laws of Addition and Subtraction

9.1 Axioms Concerning Addition, Properties of Groups

We now turn to the second group of axioms, which consists of the following six sentences (universal quantifiers are, as usual, excluded in the presentation of each axiom unless existentially quantified variables are also required in said axiom):

Axiom 6. $\forall y, z \; \exists x \; (x = y + z)$

Axiom 7. $x + y = y + x$

Axiom 8. $x + (y + z) = (x + y) + z$

Axiom 9. $\forall x, y \; \exists z \; (x = y + z)$

Axiom 10. $y < z \rightarrow x + y < x + z$

Axiom 11. $y > z \rightarrow x + y > x + z$

For the moment, let us concentrate on the first four sentences of this second group, that is, Axioms 6-9. They ascribe to the operation of addition a number of simple properties which are also frequently met when considering other operations in various parts of logic and mathematics. Special terms have been introduced to designate these properties.

Logic, New New Edition.
Karl Frinkle.

Definition 9.1. The operation O is *performable in the class* K or that the class K *is closed under the operation* O, if the performance of the operation O on any two elements of the class K results again in an element of that same class:

$$\forall\, y, z\; \exists\, x\; (x = y\, O\, z)$$

Definition 9.2. The operation O is called *commutative in the class* K if the result of this operation is independent of the order of elements of the class K on which it is carried out, or, in other words:

$$\forall\, x, y\; (x\, O\, y = y\, O\, x)$$

Definition 9.3. The operation O is called *associative in the class* K if the result of this operation is independent of the way in which elements are grouped together:

$$\forall\, x, y, z\; (x\, O\, (y\, O\, z)\,) = (\,(x\, O\, y)\, O\, z)$$

Definition 9.4. The operation O is called *right-invertible in the class* K if:

$$\forall\, x, y,\; \exists\, z\; (x = y\, O\, z)$$

Definition 9.5. The operation O is called *left-invertible in the class* K if:

$$\forall\, x, y,\; \exists\, z\; (x = z\, O\, y)$$

Definition 9.6. The operation O is called *invertible in the class* K if it is both left- and right-invertible.

It follows that if an operation O is commutative and also left- or right-invertible, it must be invertible.

Definition 9.7. A class K is a *group with respect to the operation* O if the operation O is performable, associative, and invertible in the class K.

Definition 9.8. A class K is an *Abelian group with respect to the operation* O, if K is a *group with respect to the operation* O and additionally, the operation O is commutative in the class K.

The concept of a group and, in particular, that of an Abelian group, forms the subject of a special mathematical discipline known as the *Theory of Groups*, which has already been mentioned in Chapter 5.[1]

In the case that the class K is the universal set U, we usually omit the reference to the class when employing such terms as "performable", "commutative", and so on.

In accordance with the terminology introduced above, the Axioms 6-9 are referred to as the *Law of Performability*, the *Commutative Law*, the *Associative Law*, and the *Law of Right-Invertibility* for the operation of addition, respectively; together they yield the following statement:

The set of all numbers constitutes an Abelian group with respect to addition.

9.2 Further Commutative and Associative Laws

Axiom 7, (*the Commutative Law*), and Axiom 8, (*The Associative Law*), in the form in which they have been stated here, refer to two and three numbers, respectively. But there are infinitely many other commutative and associative laws concerning more than two or three numbers. The formula

$$x + (y + z) = y + (z + x),$$

for instance, constitutes an example of a commutative law for three summands, and the formula:

$$x + [y + (z + u)] = [(x + y) + z] + u$$

represents one of many associative laws for four summands. In addition, there are theorems of a mixed character which, generally expressed, assert that any changes in either the order or the grouping of the summands are without influence upon the result of the addition. By way of an example the following theorem may be stated, and subsequently proven.

Theorem 9. $x + (y + z) = (x + z) + y$

Proof.

(1)	$z + y = y + z$	Instance of Axiom 7, $x : z$
(2)	$x + (y + z) = x + (y + z)$	Law of Reflexive Identity, (Law 3.2), $x : x + (y + z)$
(3)	$x + (y + z) = x + (z + y)$	Rule of Replacement: (1) into RHS of (2)
(4)	$x + (z + y) = (x + z) + y$	Instance of Axiom 8, $y : z$, $z : y$
(5)	$x + (y + z) = (x + z) + y$	Rule of Replacement: (4) into RHS of (3)

□

In a similar manner we can derive all commutative and associative laws concerning an arbitrary number of summands from Axioms 7 and 8 together, possibly, with Axiom 6. These theorems are often used in practice in the transformation of algebraic expressions. By a transformation of an expression denoting a number we mean, as usual, an alteration of such a kind as to lead to an expression denoting the same number, which may hence be joined with the original expression by the identity sign; the expressions most frequently subjected to transformations of this kind are those which contain variables and which, therefore, are designatory functions. On the basis of the commutative and associative laws we are in a position to transform any expressions of a form such as:

$$x + 2,\ x + (y + z),\ x + [y + (z + u)],\ \ldots$$

that is, expressions consisting of numerical constants and variables separated by addition signs and parentheses; in any such expression we may interchange at will both the numerical symbols and the parentheses (providing only the resulting expression has not become meaningless on account of the transposition of the parentheses).

9.3 Laws of Monotony For Addition and Their Converses

Axioms 10 and 11, to which we will turn now, are the so-called *Laws of Monotony* for addition with respect to the relations *less than* and *greater than.*

Definition 9.9. The binary operation O is *right-monotonic in the class* K *with respect to the two-termed relation* R if, for any elements $x, y, z \in K$, we have:

$$yRz \to (x\,O\,y)\,R\,(x\,O\,z).$$

The property of being right-monotonic means that the result of performing the operation O on x and y has the relation R to the result of performing the operation O on x and z. A similar definition can be made for left-monotonic:

Definition 9.10. The binary operation O is *left-monotonic in the class* K *with respect to the two-termed relation* R if, for any elements $x, y, z \in K$, we have:

$$yRz \to (y\,O\,x)\,R\,(z\,O\,x).$$

Similar to invertibility, if an operation O is both left- and right-monotonic, it is said to be monotonic. Furthermore, if an operation O is commutative then monotony follows from the operation being either left- or right-monotonic.

The operation of addition is monotonic not only with respect to the relations *less than* and *greater than* – a consequence of Axioms 10 and 11 – but also with respect to the other relations among numbers discussed in Section 8.4. We shall show this here only for the relation of identity:

Theorem 10. $y = z \to x + y = x + z$

Proof. This proof is relatively straightforward, but we must actually invoke Axiom 6 ($x + y \in \mathbb{R}$), as the only assumption we have from the hypothesis of this theorem is $y = z$, and it states nothing about the universally quantified variable x.

(1) $y = z$ Assume hypothesis

(2) $w = x + y \in \mathbb{R}$ Instance of Axiom 6, $x : w,\ y : x,\ z : y$

(3) $w = w$ Law of Reflexive Identity, (Law 3.2), x:w

(4) $x + y = x + y$ Rule of Replacement: (2) into LHS and RHS of (3)

(5) $x + y = x + z$ Rule of Replacement: (1) into RHS of (4)

□

The converse of Theorem 10 is also true, but before it is proven, we introduce a law which we will make use of in the proof. This law, commonly known as Modus Tollens, will be referred to as the Law of General Contradiction for the remainder of this text.

Law 9.1 (Law of General Contradiction)**.** $[\,(p \to q) \wedge \sim q\,] \to \sim p$

Theorem 11. $x + y = x + z \to y = z$

Proof. We shall sketch two proofs of this theorem here. The first is based upon the law of trichotomy and Axioms 6, 10, and 11, is comparatively simple.

(1) $x + y = x + z$ Assume hypothesis

(2) $y \neq z$ Secondary assumption

(3) $y \neq z \to y < z \vee y > z$ Instance of Axiom 1, $x : y,\ y : z$

(4) $y < z \vee y > z$ Rule of Detachment on (2) and (3)

(5) $y \not< z$ Secondary assumption

(6) $(y < z \vee y > z) \leftrightarrow (y \not< z \to y > z)$ Instance of $p \vee q \leftrightarrow \sim p \to q$ $p : y < z, q : y > z$

(7) $y \not< z \to y > z$ Substitute (6) into (4)

(8) $y > z$ Rule of Detachment on (5) and (7)

(9) $y > z \to x + y > x + z$ Axiom 11

(10) $x + y > x + z$ Rule of Detachment on (8) and (9)

(11) $x + y = x + z \to x + y \not> x + z$ Instance of Strong Law of Trichotomy, $x : x + y,\ y : x + z$

(12) $x + y \not> x + z$ Rule of Detachment on (10) and (1)

(13)	$(y > z \rightarrow x + y > x + z)$ $\wedge\, x + y \not> x + z$	Rule of And on (9) and (12)
(14)	$[\,(y > z \rightarrow x + y > x + z)$ $\wedge\, x + y \not> x + z\,] \rightarrow y \not> z$	Instance of Law 9.1, $p : y > z$, $q : x + y > x + z$
(15)	$y \not> z$	Rule of Detachment on (13) and (14)
(16)	$y \not> z \rightarrow y < z$	Contrapositive to (7)
(17)	$y < z$	Rule of Detachment on (15) and (16)
(18)	$y < z \rightarrow x + y < x + z$	Axiom 10
(19)	$x + y < x + z$	Rule of Detachment on (17) and (18)
(20)	$x + y = x + z \rightarrow x + y \not< x + z$	Instance of Strong Law of Trichotomy, $x : x + y$, $y : x + z$
(21)	$x + y \not< x + z$	Rule of Detachment on (19) and (20)
(22)	$(y < z \rightarrow x + y < x + z)$ $\wedge\, x + y \not< x + z$	Rule of And on (18) and (21)
(23)	$[\,(y < z \rightarrow x + y < x + z)$ $\wedge\, x + y \not< x + z\,] \rightarrow y \not< z$	Instance of Law 9.1, $p : y < z$, $q : x + y < x + z$
(24)	$y \not< z$	Rule of Detachment on (22) and (23)
(25)	$y \not< z \wedge y \not> z$	Rule of And on (24) and (15)
(26)	$(y \not< z \wedge y \not> z) \rightarrow y = z$	Axiom 1
(27)	$y = z$	Rule of Detachment on (25) and (26)

□

A few remarks concerning the first proof. Like the proof of Theorem 1, this first proof constitutes an example of an *indirect inference*. The schema of this proof may be represented as follows. In order to prove a certain sentence, say "p", we suppose that the sentence to be false, that is we assume "$\sim p$". From this assumption a consequence "q" is derived; that is to say, we demonstrate the implication: "$\sim p \rightarrow q$", and through the general rules of logic, we also arrive at the statement "$\sim q$", then by the law of contraposition, we also arrive at "p", which would contradict the fact that "p" was false in the first place. This combination of the Law of Contradiction and the Law of Contraposition is what we have here named the Law of General Contradiction. In the above proof, our first assumption we wished to disprove was (2) $y \neq z$. If this was assumed, then by the Law of Trichotomy, we had either $y < z$ or $y > z$, and by the Strong Law of Trichotomy, only one of these resulting relations between y and z could hold. In (4), we assumed that $y < z$ did not hold, i.e. $y \not< z$. Thus our only option for the relation between y and z is (8) $y > z$. But by (10) and our original assumption (1), we arrive, first, at a contradiction to (8), hence we know $y \not> z$. But based on this, and using (4), we now must have $y < z$ (note that this is still under the umbrella of our first secondary assumption $y \neq z$). Applying similar techniques as in the disproving $y > z$, we arrive again at a contradiction (19) to our original assumption (1) by way of the Strong Law of Trichotomy. Thus we now have (24) $y \not< z$. Once again, by the Strong Law of Trichotomy, we must conclude that $y = z$, which is contrary to our secondary assumption of $y \neq z$.

As just described, it should be apparent that the proof under consideration differs from that of Theorem 1. There, from the assumption that the theorem is false, we inferred that the theorem is true, that is, we derived a consequence directly contradicting the assumption; here however, we derived from a similar assumption a consequence of which we knew from other sources was false, namely that $x+y = z+z$ and $x+y > x+z$. This difference is not an essential one; it can easily be seen on the basis of logical laws that the proof of Theorem 1, like any other indirect mode of inference, can be brought under the schema sketched above.

For our later aims we require, however, another proof which is considerably more involved, but does not make use of anything except Axioms 7-9.

Proof.

(1)	$x + y = x + z$	Assume hypothesis
(2)	$y = y + u$	Instance of Axiom 9, $x : y$, $z : u$
(3)	$y + u = u + y$	Instance of Axiom 7, $x : y$, $y : u$
(4)	$y = u + y$	Rule of Replacement (3) into (3)
(5)	$z = y + v$	Instance of Axiom 9, $x : z$, $y : v$
(6)	$z = (u + y) + v$	Rule of Replacement (4) into (5)
(7)	$u + (y + v) = (u + y) + v$	Instance of Axiom 8, $x : u$, $v : z$
(8)	$z = u + (y + v)$	Rule of Replacement (7) into (6)
(9)	$z = u + z$	Rule of Replacement (5) into (8)
(10)	$u = x + w$	Instance of Axiom 9, $x : u$, $y : x$, $z : w$

(11)	$x + w = w + x$	Instance of Axiom 7, $y : w$
(12)	$u = w + x$	Rule of Replacement (11) into (10)
(13)	$y = (w + x) + y$	Rule of Replacement (12) into (4)
(14)	$w + (x + y) = (w + x) + y$	Instance of Axiom 8, $x : w$, $y : x$, $z : y$
(15)	$y = w + (x + y)$	Rule of Replacement (14) into (13)
(16)	$y = w + (x + z)$	Rule of Replacement (1) into (15)
(17)	$w + (x + z) = (w + x) + z$	Instance of Axiom 8, $x : w$, $y : x$
(18)	$y = (w + x) + z$	Rule of Replacement (17) into (16)
(19)	$y = u + z$	Rule of Replacement (12) into (18)
(20)	$y = z$	Rule of Replacement (9) into (20)

□

Like Theorem 10, the other laws of monotony, that is, Axioms 10 and 11, also admit of conversion:

Theorem 12. $x + y < x + z \rightarrow y < z$

Theorem 13. $x + y > x + z \rightarrow y > z$

The proof of these theorems can without difficulty be obtained along the lines of the proof of Theorem 10.

9.4 Closed Systems of Sentences

There exists a general logical law the knowledge of which considerably simplifies the proofs of Theorems 11, 12, and 13. This law, sometimes called the *Law of Closed Systems* or *Hauber's Law*[2], permits us in some cases, when we have succeeded in proving several conditional sentences, to infer from the form of these sentences that the corresponding converse sentences may be also considered as proved.

Suppose we are given a number of implications, say three, to which we will give the following schematic form:

$$p_1 \rightarrow q_1, \quad p_2 \rightarrow q_2, \quad p_3 \rightarrow q_3$$

These three sentences are said to form a *closed system*, if their antecedents are of such a kind to exhaust all possible cases, that is, if it is true that:

$$p_1 \vee p_2 \vee p_3,$$

and if at the same time, their consequents exclude one another:

$$q_1 \to \sim q_2, \quad q_1 \to \sim q_3, \quad q_2 \to \sim q_3.$$

The *Law of Closed Systems* asserts that if certain conditional sentences forming a closed system are true, then the corresponding converse sentences are also true.

The simplest example of a closed system is given in the form of a system of two sentences, consisting of some implication and its inverse:

$$p \to q, \quad \sim p \to \sim q.$$

In order to demonstrate the two converse sentences in this case, it is not even necessary to resort to the Law of Closed Systems; it is sufficient to apply the laws of contraposition to arrive at the converse statements to those given above, as the converse to $p \to q$ is the contrapositive of $\sim p \to \sim q$, and similarly, the converse to $\sim p \to \sim q$ is $\sim q \to \sim p$, which is the contrapositive of $p \to q$.

To prove the general two statement case, we assume that:

$$p_1 \to q_1, \quad p_2 \to q_2, \quad p_1 \vee p_2, \quad q_1 \to \sim q_2$$

and we wish to prove both:

$$q_1 \to p_1, \quad q_2 \to p_2$$

Proof.

(1)	$p_1 \to q_1$	Assumption
(2)	$p_2 \to q_2$	Assumption
(3)	$p_1 \vee p_2$	Assumption
(4)	$q_1 \to \sim q_2$	Assumption
(5)	$\sim q_2 \to \sim p_2$	Law of Contraposition on (2)
(6)	$(q_1 \to \sim q_2) \wedge (\sim q_2 \to \sim p_2)$	Rule of And on (4) and (5)
(7)	$[(q_1 \to \sim q_2) \wedge (\sim q_2 \to \sim p_2)]$ $\to (q_1 \to \sim p_2)$	Instance of the Law of Hypothetical Syllogism, $p : q_1$, $q :\sim q_2$, $r :\sim p_2$
(8)	$q_1 \to \sim p_2$	Rule of Detachment on (7) and (8)
(9)	$(\sim p_2 \to p_1) \leftrightarrow (p_1 \vee p_2)$	Instance of $\sim p \to q \leftrightarrow q \vee p$ $p : p_2$, $q : p_1$
(10)	$\sim p_2 \to p_1$	Rule of Substitution, with (9) into (3)

(11)	$(q_1 \rightarrow \sim p_2) \wedge (\sim p_2 \rightarrow p_1)$	Rule of And on (8) and (10)
(12)	$[(q_1 \rightarrow \sim p_2) \wedge (\sim p_2 \rightarrow p_1)] \rightarrow (q_1 \rightarrow p_1)$	Instance of the Law of Hypothetical Syllogism, $p : q_1$, $q :\sim p_2$, $r : p_1$
(13)	$q_1 \rightarrow p_1$	Rule of Detachment on (11) and (12)

Obviously the proof of $q_2 \rightarrow p_2$ is remarkably similar and will thus be omitted. We have shown that under the assumptions of a closed system the converse statements to the original conditional statements are also true. The proof of the general formulation for a closed system consisting of three conditional sentences is similar, although requires a little more work and is left as an exercise. □

Theorem 10 and Axioms 10 and 11 form a closed system of three sentences. This is a consequence of the Law of Trichotomy; since between any two numbers we have exactly one of the relations $=$, $<$, and $>$, we see that the hypotheses of these three sentences, that is, the formulas:

$$y = z, \quad y < z, \quad y > z,$$

exhaust all possible cases, while their conclusions, that is, the formulas:

$$x + y = x + z, \quad x + y < x + z, \quad x + y > x + z,$$

exclude one another. (The Law of Trichotomy implies even more, which however is irrelevant for our purpose, namely, that the first three formulas do not only exhaust all possible cases but also exclude each other, and that the last three formulas do not only exclude each other but also exhaust all possible cases.) For the mere reason that the three statements form a closed system it is true that the converse statements of Theorems 11-13 must hold.

Numerous examples of closed systems can be found in elementary geometry; for instance, when examining the relative position of two circles, we have to deal with a closed system consisting of five sentences.

In conclusion it may be remarked that anyone who does not know the law of closed systems but tries to prove the converse of statements forming a system of this kind may mechanically apply the same mode of inference which we employed in the first proof of Theorem 11.

9.5 Consequences of the Laws of Monotony

Theorems 10 and 11 may be combined into one sentence:

$$y = z \leftrightarrow x + y = x + z.$$

Similarly it is possible to combine Axioms 10 and 11 with Theorems 12 and 13. The theorems thus obtained may be denoted as *Laws of Equivalent Transformation of Equations and Inequalities* by means of addition. The content of these theorems is sometimes described as follows: if the same number is added to both sides of an equation or inequality, without changing the equality or inequality sign, the resulting equation or inequality is equivalent to the original one (this formulation is, of course, not quite correct since the sides of an equation or inequality are not numbers but expressions, to which it is not possible to add any numbers). The theorems mentioned here play an important role in the solution of equations and inequalities.

We will derive one more result from the theorems of monotony:

Theorem 14. $x + z < y + t \rightarrow (x < y \vee z < t)$

Proof. The proof of this theorem is incredibly long and tedious, and we will approach it by assuming the hypothesis and the negation of the conclusion to arrive at a contradiction.

(1)	$x + z < y + t$	Assumption
(2)	$\sim (x < y \vee z < t)$	Secondary assumption
(3)	$(x \not< y \wedge z \not< t)$ $\leftrightarrow \sim (x < y \vee z < t)$	De Morgan's Law
(4)	$x \not< y \wedge z \not< t$	Substitute (3) into (2)
(5)	$x \not< y \wedge z \not< t \rightarrow x \not< y$	Law of And, Breaking Apart
(6)	$x \not< y \wedge z \not< t \rightarrow z \not< t$	Law of And, Breaking Apart
(7)	$x \not< y$	Rule of Detachment on (4) and (5)
(8)	$z \not< t$	Rule of Detachment on (4) and (6)
(9)	$x \not< y \leftrightarrow (x = y \vee x > y)$	Definition of $\not<$
(10)	$z \not< t \leftrightarrow (z = t \vee z > t)$	Definition of $\not<$
(11)	$x = y \vee x > y$	Substitute (9) into (7)
(12)	$z = t \vee z > t$	Substitute (10) into (8)
(13)	$(x = y \vee x > y) \wedge (z = t \vee z > t)$	Rule of And, Joining Together on (11) and (12)
(14)	$(x = y \vee x > y) \wedge (z = t \vee z > t)$ $\leftrightarrow \{[(x = y \vee x > y) \wedge z = t] \vee$	Distributive Law on (13)

$[(x = y \vee x > y) \wedge z > t]\}$

(15)	$[(x = y \vee x > y) \wedge z = t] \vee$ $[(x = y \vee x > y) \wedge z > t]$	Substitute (14) into (13)
(16)	$[(x = y \vee x > y) \wedge z = t]$ $\leftrightarrow \{(x = y \wedge z = t)$ $\vee (x > y \wedge z = t)\}$	Distributive Law
(17)	$[(x = y \vee x > y) \wedge z > t]$ $\leftrightarrow \{(x = y \wedge z > t)$ $\vee (x > y \wedge z > t)\}$	Distributive Law
(18)	$(x = y \wedge z = t) \vee$ $(x > y \wedge z = t) \vee$ $(x = y \wedge z > t) \vee$ $(x > y \wedge z > t)$	Substitute (16), (17) into (15)
(19)	$x = y \wedge z = t$	Secondary assumption
(20)	$x = y \wedge z = t \rightarrow x = y$	Law of And, Breaking Apart
(21)	$x = y \wedge z = t \rightarrow z = t$	Law of And, Breaking Apart
(22)	$x = y$	Rule of Detachment, (19) and (20)
(23)	$z = t$	Rule of Detachment, (19) and (21)
(24)	$x = y \rightarrow z + x = z + y$	Instance of Theorem 10 $x : z,\ y : x,\ z : y$
(25)	$z + x = z + y$	Rule of Detachment, (22) and (24)
(26)	$z + x = x + z$	Instance of Axiom 7, $x : z$, $y : x$
(27)	$x + z = z + y$	Rule of Replacement, (26) into (25)
(28)	$z + y = y + z$	Instance of Axiom 7, $x : z$
(29)	$x + z = y + z$	Rule of Replacement,(28) into (27)
(30)	$x + z = y + t$	Rule of Replacement, (23) into RHS of (29)
(31)	$x + z = y + t \rightarrow x + z \not< y + t$	Law of Trichotomy

(32)	$x + z \not< y + t$	Rule of Detachment, (30), (31)
(33)	$[(x = y \wedge z = t \to x + z \not< y + t)$ $\wedge (x + z < y + t)]$ $\to \sim (x = y \wedge z = t)$	Law of General Contradiction
(34)	$x = y \wedge z = t \to x + z \not< y + t$	Instance of $\text{True}_1 \to \text{True}_2$ $\text{True}_1 : x + z = y + t$ $\text{True}_2 : x + z \not< y + t$
(35)	$[(x = y \wedge z = t \to x + z \not< y + t)$ $\wedge (x + z < y + t)]$	Rule of And, Joining Together on (1) and (37)
(36)	$\sim (x = y \wedge z = t)$	Rule of Detachment, (33) and (35) (Contradiction to (19))
(37)	$x > y \wedge z = t$	Secondary assumption
(38)	$x > y \wedge z = t \to x > y$	Law of And, Breaking Apart
(39)	$x > y \wedge z = t \to z = t$	Law of And, Breaking Apart
(40)	$x > y$	Rule of Detachment, (37) and (38)
(41)	$z = t$	Rule of Detachment, (37) and (39)
(42)	$x > y \to z + x > z + y$	Instance of Axiom 11 $x : z,\ y : x,\ z : y$
(43)	$z + x > z + y$	Rule of Detachment, (40) and (42)
(44)	$z + x = x + z$	Instance of Axiom 7, $x : z$, $y : x$
(45)	$x + z > z + y$	Rule of Replacement,(44) into (43)
(46)	$z + y = y + z$	Instance of Axiom 7, $x : z$
(47)	$x + z > y + z$	Rule of Replacement,(46) into (45)
(48)	$x + z > y + t$	Rule of Replacement, (41) into RHS of (47)
(49)	$x + z > y + t \to x + z \not< y + t$	Law of Trichotomy
(50)	$x + z \not< y + t$	Rule of Detachment, (48), (49)
(51)	$[(x > y \wedge z = t \to x + z \not< y + t)$	Law of General

$\wedge\,(x+z<y+t)\,]$ Contradiction
$\to\sim(x>y\wedge z=t)$

(52) $x>y\wedge z=t\to x+z\not<y+t$ Instance of $\text{True}_1\to\text{True}_2$
$\text{True}_1: x>y\wedge z=t$
$\text{True}_2: x+z\not<y+t$

(53) $[\,(x>y\wedge z=t\to x+z\not<y+t)$ Rule of And, Joining Together
$\wedge\,(x+z<y+t)\,]$ on (1) and (52)

(54) $\sim(x>y\wedge z=t)$ Rule of Detachment, (51) and (53)
(Contradiction to (37))

(55) $x=y\wedge z>t$ Secondary assumption

(56) $x=y\wedge z>t\to x=y$ Law of And, Breaking Apart

(57) $x=y\wedge z>t\to z>t$ Law of And, Breaking Apart

(58) $x=y$ Rule of Detachment, (55) and (56)

(59) $z>t$ Rule of Detachment, (55) and (57)

(60) $z>t\to x+z>x+t$ Instance of Axiom 7
$x:x,\ y:z,\ z:t$

(61) $x+z>x+t$ Rule of Detachment, (59) and (60)

(62) $x+z>y+t$ Rule of Replacement,
(58) into RHS of (61)

(63) $x+z>y+t\to x+z\not<y+t$ Law of Trichotomy

(64) $x+z\not<y+t$ Rule of Detachment, (62), (63)

(65) $[\,(x=y\wedge z>t\to x+z\not<y+t)$ Law of General
$\wedge\,(x+z<y+t)\,]$ Contradiction
$\to\sim(x=y\wedge z>t)$

(66) $x=y\wedge z>t\to x+z\not<y+t$ Instance of $\text{True}_1\to\text{True}_2$
$\text{True}_1: x=y\wedge z>t$
$\text{True}_2: x+z\not<y+t$

(67) $[\,(x=y\wedge z>t\to x+z\not<y+t)$ Rule of And, Joining Together
$\wedge\,(x+z<y+t)\,]$ on (1) and (66)

(68)	$\sim (x = y \wedge z > t)$	Rule of Detachment, (65) and (67) (Contradiction to (55))
(69)	$x > y \wedge z > t$	Secondary assumption
(70)	$x > y \wedge z > t \rightarrow x > y$	Law of And, Breaking Apart
(71)	$x > y \wedge z > t \rightarrow z > t$	Law of And, Breaking Apart
(72)	$x > y$	Rule of Detachment, (55) and (56)
(73)	$z > t$	Rule of Detachment, (55) and (57)
(74)	$z > t \rightarrow y + z > y + t$	Instance of Axiom 7 $x : y$, $y : z$, $z : t$
(75)	$x > y \rightarrow z + x > z + y$	Instance of Axiom 7 $x : z$, $y : x$, $z : y$
(76)	$y + z > y + t$	Rule of Detachment, (73) and (74)
(77)	$z + x > z + y$	Rule of Detachment, (72) and (75)
(78)	$z + x = x + z$	Instance of Axiom 7, $x : y$, $y : x$
(79)	$x + z > z + y$	Rule of Replacement,(78) into (77)
(80)	$y + z = z + y$	Instance of Axiom 7, $x : y$, $z : y$
(81)	$z + y > y + t$	Rule of Replacement, (80) into (76)
(82)	$x + z > z + y \wedge z + y > y + t$	Rule of And, Joining Together on (79) and (81)
(83)	$x + z > z + y \wedge z + y > y + t$ $\rightarrow x + z > y + t$	Instance of Axiom 5, $x : x + z$, $y : z + y$, $z : y + t$
(84)	$x + z > y + t$	Rule of Detachment, (82) and (83)
(85)	$x + z > y + t \rightarrow x + z \not< y + t$	Law of Trichotomy
(86)	$x + z \not< y + t$	Rule of Detachment, (84), (85)
(87)	$[(x > y \wedge z > t \rightarrow x + z \not< y + t)$ $\wedge (x + z < y + t)]$ $\rightarrow \sim (x > y \wedge z > t)$	Law of General Contradiction
(88)	$x > y \wedge z > t \rightarrow x + z \not< y + t$	Instance of $\text{True}_1 \rightarrow \text{True}_2$

		$\text{True}_1 : x > y \wedge z > t$ $\text{True}_2 : x + z \not< y + t$
(89)	$[(x > y \wedge z > t \to x + z \not< y + t)$ $\wedge (x + z < y + t)]$	Rule of And, Joining Together on (1) and (88)
(90)	$\sim (x > y \wedge z > t)$	Rule of Detachment, (87) and (89) (Contradiction to (69))
(91)	$\sim (x = y \wedge z = t) \wedge$ $\sim (x > y \wedge z = t) \wedge$ $\sim (x = y \wedge z > t) \wedge$ $\sim (x > y \wedge z > t)$	Rule of And, Joining Together on (36), (54), (68), (90)
(92)	$\sim [(x = y \wedge z = t)$ $\vee (x > y \wedge z = t)$ $\vee (x = y \wedge z > t)$ $\vee (x > y \wedge z > t)]$	De Morgan's Law on (91)
(93)	$(92) \leftrightarrow \sim (18)$	Definition of $\sim$
(94)	$\sim (18)$	Substitute (93) into (92)
(95)	$\sim (x < y \vee z < t) \to (18)$	Instance of $\text{True}_1 \to \text{True}_2$ $\text{True}_1 : \sim (x < y \vee z < t)$ $\text{True}_2 : (18)$
(96)	$[\sim (x < y \vee z < t) \to (18)$ $\wedge \sim (18)]$	Rule of And, Joining Together on (95) and (94)
(97)	$[\sim (x < y \vee z < t) \to (18)$ $\wedge \sim (18)]$ $\to (x < y \vee z < t)$	Law of General Contradiction
(98)	$x < y \vee z < t$	Rule of Detachment, (96) and (97)

As was originally stated, the proof was long and tedious, so we will try to emphasize some of the more important pieces of the proof. By assuming the hypothesis and not the conclusion, we hope to arrive at a contradiction. The negation of the conclusion: (2) $\sim (x < y \vee z < t)$, will result in four possible pairs of relations between x and y, and z and t, which is given in (18), hence all steps between (3) and (17) are simply deriving this fact. Once we have all four of these possibilities, we simply go through each one and derive a contradiction. In (19) we assume $x = y \wedge z = t$ and arrive at the statement: (30) $x + z = y + t$, which is a direct contradiction to our hypothesis (by the Law of Trichotomy). Next up, is (37) $x > y \wedge z = t$, for which arrive at the statement: (48) $x + z > y + t$, once again a direct contradiction to the hypothesis. The third pair of relations is given by (55) $x = y \wedge z > t$ which, in a similar set of steps to the previous pair, arrives at the same contradictory relation to the hypothesis of (62) $x + z > y + t$. The last pair of relations to assume is (69) $x > y \wedge z > t$, and to arrive at a contradiction, the transitivity property of the relation $>$ was used, and (84) $x + z > y + t$ is the same contradiction as the previous two pairs of assumed relations.

Once all four possible pairs of relations have been contradicted, their negations are joined by the Rule of And, Joining Together. These form the negation of the statement in (18), which is the last statement prior to our assumptions starting in (19). Thus, the assumption used to arrive at all statements before (19) are false, which means the secondary assumption (2) $\sim (x < y \vee z < t)$ is false as well. Thus $x < y \vee z < t$, which is exactly what we were trying to prove, assuming the hypothesis. □

The argument just conducted is counted among the indirect proofs; apart from an inessential modification, it could be brought under the schema sketched in Section 9.3 in connection with the first proof of Theorem 11. Formally considered, however, the procedure of the argument is slightly different from the one followed in the proofs of Theorems 1 and 11. The inference has the following schema. In order to prove a conditional sentence $p \rightarrow q$, we assume the conclusion of the sentence, that is "q", to be false (and not the whole sentence); from this assumption, that is, from "$\sim q$", it is inferred that the hypothesis is false, that is, that "$\sim p$" holds. In other words, instead of demonstrating the sentence in question, a proof of the contrapositive sentence is given. Note that in the details of the above proof, we arrived, four times, at "$\sim (x + z < y + t)$" to the hypothesis. Inferences of this form are very common in all mathematical disciplines; they constitute the most usual type of indirect proof.

9.6 Definition of Subtraction; Inverse Operations

Our next task is to show how the notion of subtraction can be introduced into our considerations. With this aim in mind, we shall first prove the following theorem:

Theorem 15. *For any two numbers y and z, there is exactly one number x such that $y = z + x$.*

Proof. The existence of an x which satisfies $y = z + x$ is guaranteed by Axiom 9. What we need to show is that if any other value u satisfies the equation, then $u = x$. We can use the quantified definition of exactly one (Equation (3.8) of Section 3.5) to prove this.

(1)	$y = z + x$	Instance of Axiom 9, $x : y,\ y : z,\ x : z$
(2)	$y = z + u$	Assumption
(3)	$z + x = z + u$	Rule of Replacement, (1) and (2)
(4)	$z + x = z + u \to x = u$	Instance of Theorem 11 $x : z,\ y : x,\ z : u$
(5)	$x = u$	Rule of Detachment on (3) and (5)

□

This unique number x, of which the above theorem treats, is designated by the symbol:

$$y - z,$$

we read it, as usual, "*the difference of the numbers x and y*" or "*the result of subtracting the number z from the number y*". The precise definition of the notion of difference is as follows:

Definition 2. $x = y - z \overset{def}{\longleftrightarrow} y = z + x$

We can generalize the idea of subtraction, as we have with other operations of arithmetic, and introduce the definitions of inverse operations.

Definition 9.11. An operation I is called a *right inverse of the operation O in the class K* if the two operations O and I fulfill the following condition:

$$\forall\, x, y, z\ (x = y\, I\, z \leftrightarrow y = z\, O\, x)$$

Definition 9.12. An operation I is called a *left inverse of the operation O in the class K* if the two operations O and I fulfill the following condition:

$$\forall\, x, y, z\ (x = y\, I\, z \leftrightarrow y = x\, O\, z)$$

If the operation O is commutative in the class K, its two inverses, the right and the left, coincide, and we can then simply speak of the *inverse of the operation O* (or, also, of the *inverse operation of O*). In accordance with this terminology, Definition 2 expresses the fact that subtraction is the right inverse (or simply, the inverse) of addition.

9.7 Definitions Containing Identity

Definition 2 exemplifies a kind of definition very common in mathematics. These definitions stipulate the meaning of a symbol designating either a single thing or an operation on a certain number of things (in other words, a function with a certain number of arguments). In every definition of this kind, the definiendum has the form of an equation:

$$x = \ldots;$$

on the right side of this equation, we have the symbol itself which was to be defined, or else a designatory function constructed out of the symbol to be defined and certain variables "y", "z".... The symbol in question designates a single thing or an operation on things. The definiens may be a sentential function of any form, which contains the same free variables as the definiendum, and which states that the thing x, together possibly with the things y, z, ..., satisfies such and such a condition. Definition 2 establishes the meaning of a symbol which denotes an operation on two numbers. To give a different example of this type of definition, let us state the definition of the symbol 0 which designates a single number:

Definition. $x = 0 \overset{def}{\longleftrightarrow} \forall y\ (y + x = y)$

A certain danger is connected with definitions of the type under consideration; for if one does not proceed with sufficient caution in laying down such definitions, one can easily find oneself confronted with a contradiction. A concrete example will make this clear.

Let us leave, for the moment, our present investigation, and assume that in arithmetic we have already the symbol of multiplication at our disposal and that, with its help, we want to define the symbol of division. For this purpose we proceed to lay down the following definition, which is modelled precisely after Definition 2:

Definition. $x = y/z \overset{def}{\longleftrightarrow} y = z \cdot x$

If now, in this definition, we replace both "y" and "z" by "0", and "x" first by "1" and then by "2", and if we observe that we have the formulas

$$0 = 0 \cdot 1, \quad 0 = 0 \cdot 2,$$

we obtain at once:

$$1 = 0/0, \quad 2 = 0/0.$$

But since two things equal to the same thing are equal to each other, we arrive at

$$1 = 2,$$

which is obviously nonsense.

It is not hard to exhibit the reason for this phenomenon. Both in Definition 2 and in the definition of the quotient considered here, the definiens has the form of a sentential function with three free variables "x", "y", and "z". To each such sentential function there corresponds a three-termed relation holding between the numbers x, y, and z if, and only if, these numbers satisfy that sentential function (cf. Section 5.1); and it is just the aim of the definition to introduce a symbol designating this relation. But if one gives the definiendum the form:

$$x = y - z, \quad \text{or} \quad x = y/z,$$

one assumes in advance that this relation is functional (and hence an operation, or a function, cf. Section 5.8), and that therefore, to any two numbers y and z, there is at most one number x standing to them in the relation in question. The fact that the relation is functional, however, is not at all evident from the beginning, and it must first be established. This we did in the case of Definition 2; but we failed to do so in the case of the definition of the quotient, and we would indeed have been unable to do so, simply because the relation in question ceases to be functional in a certain exceptional case: for if $y = 0$ and $z = 0$, there exists infinitely many numbers x for which $y = z \cdot x$. If, therefore, one wants to formulate the definition of the quotient in the above form without introducing contradictions, one has to take care that the case is excluded where both numbers y and z are 0, – for instance, by inserting an additional condition in the definiens.

The above considerations lead us to the following conclusion. Every definition of the type of Definition 2 should be preceded by a theorem corresponding exactly to Theorem 15, that is to say, a theorem to the effect that there is but one number x which satisfies the definiens. (The question arises whether it is relevant if there is exactly one number x, or whatever it is sufficient that there is at most one such number. A discussion of this rather difficult problem will be omitted here.)

9.8 Theorems on Subtraction

On the basis of Definition 2 and the laws of addition we can without difficulty prove the fundamental theorems of the *Theory of Subtraction*, such as the *Law of Performability*, the *Laws of Monotony*, and the *Laws of Equivalent Transformation of Equations and Inequalities by Means of Subtraction.* Those theorems also belong here which make possible the transformation of so-called algebraic sums, that is, of expressions consisting of numerical constants and variables, separated by "+" and "−" signs as well as parentheses (the latter often being omitted in accordance with special rules to this effect.) The following theorem may serve as an example of the last-named category:

Theorem 16. $x + (y - z) = (x + y) - z$.

Proof. The existence of an x which satisfies $y = z + x$ is guaranteed by Axiom 9. What we need to show is that if any other value u satisfies the equation, then $u = x$. We can use the quantified definition of exactly one (Equation (3.8) of Section 3.5) to prove this.

(1)	$y = z + u$	Instance of Axiom 9, $x : y,\ y : z,\ z : u$
(2)	$u = y - z \leftrightarrow y = z + u$	Instance of Definition 2, $x : u$
(3)	$u = y - z$	Rule of Replacement (2) into (1)
(4)	$x + y = y + x$	Axiom 7
(5)	$x + y = (z + u) + x$	Rule of Replacement (1) into (4)
(6)	$z + (u + x) = (z + u) + x$	Instance of Axiom 8, $x : y,\ y : u,\ z : x$
(7)	$u + x = x + u$	Instance of Axiom 7, $x : u,\ y : x$
(8)	$z + (x + u) = (z + u) + x$	Rule of Replacement (7) into LHS of (6)
(9)	$x + y = z + (x + u)$	Rule of Replacement (8) into RHS of (5)
(10)	$x + y \in \mathbb{R}$	Axiom 6
(11)	$x + u \in \mathbb{R}$	Axiom 6
(10)	$x + u = (x + y) - z$	Instance of Definition 2, $x : x + u,\ y : x + y$
(11)	$x + (y - z) = (x + y) - z$	Rule of Replacement (3) into LHS of (10)

$\square$

Having got this far, we now terminate the construction of our fragment of arithmetic. We conclude this chapter with a list of all the axioms and theorems from Chapters 8 and 9.

Axiom 1. $\forall x, y\ (x = y \lor x < y \lor x > y)$

Axiom 2. $x < y \rightarrow y \not< x$

Axiom 3. $x > y \rightarrow y \not> x$

Axiom 4. $(x < y \land y < z) \rightarrow x < z$

Axiom 5. $(x > y \wedge y > z) \rightarrow x > z$

Axiom 6. $\forall\, y, z\ \exists\, x\ (x = y + z)$

Axiom 7. $x + y = y + x$

Axiom 8. $x + (y + z) = (x + y) + z$

Axiom 9. $\forall\, x, y\ \exists\, z\ (x = y + z)$

Axiom 10. $y < z \rightarrow x + y < x + z$

Axiom 11. $y > z \rightarrow x + y > x + z$

Theorem 1. $x \not< x$

Theorem 2. $x \not> x$

Theorem 3. $x > y \longleftrightarrow y < x$

Theorem 4. $x \neq y \longleftrightarrow (x < y \vee y < x)$

Theorem 5. $x \neq y \longleftrightarrow (x > y \vee y > x)$

Theorem 6. *For any numbers $x, y \in \mathbb{R}$, exactly one of the three formulas: $x = y$, $x < y$ and $x > y$ is satisfied.*

Theorem 7. $x \leq y \leftrightarrow x \not> y$

Theorem 8. $x < y \leftrightarrow (x \leq y \wedge x \neq y)$

Theorem 9. $x + (y + z) = (x + z) + y$

Theorem 10. $y = z \rightarrow x + y = x + z$

Theorem 11. $x + y = x + z \rightarrow y = z$

Theorem 12. $x + y < x + z \rightarrow y < z$

Theorem 13. $x + y > x + z \rightarrow y > z$

Theorem 14. $x + z < y + t \rightarrow (x < z \vee z < t)$

Theorem 15. *For any two numbers y and z, there is exactly one number x such that $y = z + x$.*

Theorem 16. $x + (y - z) = (x + y) - z.$

Exercises

1. Consider the following three systems, each consisting of a certain set, two relations, and one operation. Determine which of these systems are models of the system of Axioms 1-11 (cf. Section 6.2).

(a) The set of all numbers, the relations $\leq$, $\geq$, the operation of addition.

(b) The set of all numbers, the relations $<$, $>$, the operation of multiplication.

(c) The set of all positive numbers, the relations $<$, $>$, the operation of multiplication.

2. Let us consider four operations A, B, G, and L which, like addition, correlate a third number with any two numbers. As the result of this operation A on numbers x and y we always consider the number x, and as a result of the operation B, we always consider the number y:

$$x\,A\,y = x, \quad x\,B\,y = y.$$

By the symbols $x\,G\,y$ and $x\,L\,y$ we denote that of the two numbers x and y which is not less than or not greater than the other, respectively; we thus have:

$$x\,G\,y = \max(x, y), \quad x\,L\,y = \min(x, y).$$

Note: you may assume that $\max(x, x) = \min(x, x) = x$.

(a) Which of the properties discussed in Section 9.1 belong to these four operations?

(b) Is the set of all numbers a group and, in particular, an Abelian group with respect to any of these operations?

3. Show that the set of all numbers is not an Abelian group with respect to multiplication.

4. Show that the following sets are all Abelian groups with respect to multiplication.

(a) The set of all numbers different from 0.

(b) The set of all positive numbers.

(c) The set consisting of the two numbers 1 and -1.

5. Consider the set $\mathbb{Z}_2 = \{0, 1\}$ (i.e. the set consisting solely of the numbers 0 and 1), and let the operation $\overset{2}{+}$ on the elements of this set be defined by the following formulas:

$$0 \overset{2}{+} 0 = 1 \overset{2}{+} 1 = 0,$$
$$0 \overset{2}{+} 1 = 1 \overset{2}{+} 0 = 1.$$

Determine whether or not the set $\mathbb{Z}_2$ is an Abelian group with respect to the operation $\overset{2}{+}$.

6. Consider the set $\mathbb{Z}_3 = \{0, 1, 2\}$. Define an operation $\overset{3}{+}$ on the elements of this set so that the set $\mathbb{Z}_3$ will be an Abelian group with respect to this operation.

7. Prove that no set consisting of two or three different numbers can be an Abelian group with respect to the standard definition of addition.

8. Is there a set consisting of one single number that forms an Abelian group with respect to addition?

9. Derive the following theorems from Axioms 6-8:

Theorem A. $x + (y + z) = (z + x) + y$

Theorem B. $x + [\,y + (z + t)\,] = (t + y) + (x + z)$

10. How many expressions can be obtained from each of these expressions:

$$x + (y + z), \quad x + [\,y + (z + t)\,], \quad x + \{y + [\,z + (t + u)\,]\,\}$$

if they are transformed solely on the basis of Axioms 6-8?

11. On the basis of the axioms adopted by us and the theorems derived from them, prove that addition is a monotonic operation with respect to the relations:
(a) $\neq$ (b) $\leq$ (c) $\geq$

12. Determine if multiplication is a monotonic operation with respect to the relations $<$ and $>$ for each of the following sets:
(a) $\mathbb{R}$
(b) the set of all positive numbers
(c) the set of all negative numbers

13. Determine which of the operations defined in Exercise 2 are monotonic with respect to the relations: $=$, $<$, $>$, $\neq$, $\leq$, and $\geq$.

14. Determine if the union and intersection of classes are monotonic with respect to each of the following class relations:
(a) proper subclass
(b) disjoint
(c) overlap

(d) class equality

15. Derive from our axioms and theorems the following theorem:

Theorem C. $(x < y \wedge z < t) \rightarrow x + z < y + t$

16. Replace in the theorem of Exercise 15 the symbol $<$ in turn by each of the following symbols and examine whether the sentences obtained are true. You do not have to prove them, just provide a convincing argument.
(a) $>$ (b) $=$ (c) $\neq$ (d) $\leq$ (e) $\geq$

17. Derive the following theorems from our axioms by first proving each of the converse statements (using the results of Exercises 15 and 16) and arguing that the converse sentences form a closed system.

Theorem D. $x + x = y + y \rightarrow x = y$

Theorem E. $x + x < y + y \rightarrow x < y$

Theorem F. $x + x > y + y \rightarrow x > y$

18. Do the operations introduced in Exercise 2 possess inverses in the set $\mathbb{R}$?

19. Derive the following theorems from our axioms and Definition 2:

Theorem G. $x - (y + z) = (x - y) - z$

Theorem H. $x - (y - z) = (x - y) + z$

Theorem I. $x + y = x - [(x - y) - x]$

Notes

[1]The group concept was introduced into mathematics by the French mathematician E. Galois (1811-1832). The term "Abelian Group" was chosen in honor of the Norwegian mathematician N. H. Abel (1802-1929) whose researches have had a great influence upon the development of higher algebra. The far-reaching importance of the group concept for mathematics has been recognized particularly since the works of another Norwegian mathematician, S. Lie (1842-1899).

[2]After the name of the German mathematician K. F. Hauber (1775-1851).

Chapter 10

Methodological Considerations on the Constructed Theory

10.1 Elimination of Superfluous Axioms in the Original Axiomatic System

The two preceding chapters were devoted to an outline of the foundations of an elementary mathematical theory which constitutes a fragment of arithmetic. In the present chapter we shall proceed to considerations of a methodological nature, concerning the system of axioms and primitive terms upon which that theory is based.

We shall begin with concrete examples illustrating the remarks of Section 6.4 concerning such problems as arbitrariness in the selection of axioms and primitive terms, the possible omission of superfluous axioms, and so on.

First, we recall our current system of axioms from Chapters 8 and 9:

Axiom 1. $\forall x, y\ (x = y \lor x < y \lor x > y)$

Axiom 2. $x < y \rightarrow y \not< x$

Axiom 3. $x > y \rightarrow y \not> x$

Axiom 4. $(x < y \land y < z) \rightarrow x < z$

Axiom 5. $(x > y \land y > z) \rightarrow x > z$

Axiom 6. $\forall y, z\ \exists x\ (x = y + z)$

Axiom 7. $x + y = y + x$

Logic, New New Edition.
Karl Frinkle.

Axiom 8. $x + (y + z) = (x + y) + z$

Axiom 9. $\forall x, y \; \exists z \; (x = y + z)$

Axiom 10. $y < z \rightarrow x + y < x + z$

Axiom 11. $y > z \rightarrow x + y > x + z$

We will refer to this axiomatic system as *System* $\mathfrak{U}$. The question is whether or not System $\mathfrak{U}$ possibly contains any superfluous axioms, that is, axioms which can be derived from the remaining axioms of the system. We shall see at once that it is easy to answer this question, and moreover, affirmatively. In fact, we have the following:

Three of the axioms of System $\mathfrak{U}$*, namely, one of the Axioms 4 and 5, Axioms 6, and one of Axioms 10 and 11, can be derived from the remaining axioms.*

We break this statement down into the following three theorems:

Theorem I. *Either of the Axioms 4 or 5 can be derived from the other with the help of Axioms 1-3.*

Proof. The first thing to note is the proof of Theorem 3 was based solely, whether directly or indirectly, upon Axioms 1-3. Thus we may use Theorem 3 in this proof. We will prove that Axiom 5 can be derived, while recognizing that the derivation Axiom 4 is similar.

(1)	$(z < y \wedge y < x) \rightarrow z < x$	Instance of Axiom 4, $x : z,\ z : x$
(2)	$(y < x \wedge z < y) \leftrightarrow (z < y \wedge y < x)$	Instance of $p \wedge q \leftrightarrow q \wedge p$ $p : y < x,\ q : z < y$
(3)	$(y < x \wedge z < y) \rightarrow z < x$	Substitute (2) into (1)
(4)	$x > y \leftrightarrow y < x$	Theorem 3
(5)	$y > z \leftrightarrow z < y$	Instance of Theorem 3 $x : y,\ y : z$
(6)	$x > z \leftrightarrow z < x$	Instance of Theorem 3 $x : x,\ y : x$
(7)	$(x > y \wedge y > z) \rightarrow x > z$	Substitute (4), (5), and (6) into (3)

□

Theorem II. *Either of the Axioms 10 or 11 can be derived from the other with the help of Axioms 1-3.*

Proof. Once again, we will make use of the fact that the proof of Theorem 3 was based solely upon Axioms 1-3.

(1)	$z < y \rightarrow x + z < x + y$	Instance of Axiom 10, $y : z$, $z : y$
(2)	$x + y > x + z \leftrightarrow x + z < x + y$	Instance of Theorem 3 $x : x + y$, $y : x + z$
(3)	$y > z \leftrightarrow z < y$	Instance of Theorem 3 $x : y$, $y : z$
(4)	$y > z \rightarrow x + y > x + z$	Substitute (2) and (3) into (1)

□

Theorem III. *Axiom 6 can be derived from Axioms 7-9.*

Proof. The proof of this theorem is not as simple as that of Theorems I and II, and resembles the second proof of Theorem 11, as given in Section 9.3. Two arbitrary numbers x and y are given; by a fourfold application of Axiom 9, four new numbers u, w, z, and v are introduced and rearranged to arrive at Axiom 6.

(1)	$y = y + u$	Instance of Axiom 9, $x : y$, $z : u$
(2)	$u = x + w$	Instance of Axiom 9, $x : u$, $y : x$, $z : w$
(3)	$y = w + z$	Instance of Axiom 9, $x : y$, $y : w$
(4)	$z = y + v$	Instance of Axiom 9, $x : z$, $z : v$
(5)	$y + u = u + y$	Instance of Axiom 7, $x : y$, $y : u$
(6)	$y = u + y$	Rule of Replacement (5) into (1)
(7)	$z = (u + y) + v$	Rule of Replacement (6) into (4)

(8)	$(u + y) + v = u + (y + v)$	Instance of Axiom 8, $x : u$, $z : v$
(9)	$z = u + (y + v)$	Rule of Replacement (8) into (7)
(10)	$z = u + z$	Rule of Replacement (4) into (9)
(11)	$z = (x + w) + z$	Rule of Replacement (2) into (10)
(12)	$(x + w) + z = x + (w + z)$	Instance of Axiom 8, $y : w$
(13)	$z = x + (w + z)$	Rule of Replacement (12) into (11)
(14)	$z = x + y$	Rule of Replacement (3) into (13)

Thus, we have shown that, for any two numbers x and y, there exists a number z for which (14) above holds; which is the content of Axiom 6. □

It might be added that the mode of inference sketched above applies, not only to addition, but, in accordance with the general remarks of Sections 6.2 and 6.3, also to any other operation; as operation O which is commutative, associative, and right-invertible in a class K is also performable in that class, and the class K, therefore, forms an Abelian group with respect to the operation O (cf. Section 9.1).

We have seen now that System $\mathfrak{U}$ contains at least three axioms which are superfluous and may therefore be omitted. Consequently, System $\mathfrak{U}$ may be replaced by the system consisting of the following eight axioms:

Axiom $1^{(1)}$**.** $\forall x, y\ (x = y \vee x < y \vee x > y)$

Axiom $2^{(1)}$**.** $x < y \rightarrow y \not< x$

Axiom $3^{(1)}$**.** $x > y \rightarrow y \not> x$

Axiom $4^{(1)}$**.** $(x < y \wedge y < z) \rightarrow x < z$

Axiom $5^{(1)}$**.** $x + y = y + x$

Axiom $6^{(1)}$**.** $x + (y + z) = (x + y) + z$

Axiom $7^{(1)}$**.** $\forall x, y\ \exists z\ (x = y + z)$

Axiom $8^{(1)}$**.** $y < z \rightarrow x + y < x + z$

We shall refer to this axiomatic system as System $\mathfrak{U}^{(1)}$, and we now have the following result:

Theorem. *Systems* $\mathfrak{U}$ *and* $\mathfrak{U}^{(1)}$ *are equipollent.*

Proof. Clearly anything which can be proven in System $\mathfrak{U}^{(1)}$ can be proven in System $\mathfrak{U}$ since the axioms of System $\mathfrak{U}^{(1)}$ are a subset of the axioms of System $\mathfrak{U}$. Furthermore, after proving Theorems I-III, the converse holds. Thus the theorem is proved. □

In comparison with the original system, the new simplified axiomatic system has certain shortcomings, both from the esthetic and the didactical points of view. It is not longer symmetric with respect to the two primitive symbols "$<$" and "$>$", certain properties of the relation $<$ being accepted without proof, while quite analogous properties of the relation $>$ have first to be demonstrated. Also, in the new system, Axiom 6 is missing, which was of a very elementary and intuitively evident character, while its derivation from the axioms contained in System $\mathfrak{U}^{(1)}$ offers some difficulties (cf. the proof of Theorem III).

10.2 Independence of the Axioms of the Simplified System

The question now arises whether there are any further superfluous axioms contained in System $\mathfrak{U}^{(1)}$. It will turn out this this is not the case.

Theorem. *System $\mathfrak{U}^{(1)}$ is a system of mutually independent axioms.*

Proof. In order to establish this methodological statement, we employ the method of proof by interpretation, which has already been used in a particular case in Section 6.2.

We are to show that no axiom of System $\mathfrak{U}^{(1)}$ is derivable from the remaining axioms of the system. Let us consider, for example, Axiom $2^{(1)}$. Suppose we replace the symbol "$<$" in the axioms of System $\mathfrak{U}^{(1)}$ throughout by "$\leq$", without altering the axioms in any other way. As a result of this transformation, no axiom, with the exception of Axiom $2^{(1)}$, loses its validity; in fact, Axioms $3^{(1)}$, $5^{(1)}$, $6^{(1)}$, and $7^{(1)}$, since they do not contain the symbol "$<$", are left unaltered, and Axioms $1^{(1)}$, $4^{(1)}$, and $8^{(1)}$ go over into certain arithmetical theorems whose proofs on the basis of System $\mathfrak{U}$ or System $\mathfrak{U}^{(1)}$ and Definition 1 (the definition of the symbol "$\leq$", cf. Section 8.4) present no difficulties. It may, therefore, be asserted that the set $\mathbb{R}$ of all numbers, the relations $\leq$ and $>$, and the operation of addition, form a model of the Axioms$1^{(1)}$ and $3^{(1)}$-$8^{(1)}$; the system of these seven axioms has thus found a new interpretation within arithmetic.

On the other hand, Axiom $2^{(1)}$, with this new interpretation, looks as follows:

Axiom $2^{(1)}$ ($\leq$ Interpretation) . $x \leq y \to y \nleq x$

Remember that it is understood that x and y are universally quantified in this axiom. In particular, by setting $x = y$, we know from the definition of $\leq$ (and subsequently $\geq$) that:

$$x = y \to (x \geq y \wedge x \leq y),$$

which is a direct contradiction to the current interpretation of Axiom $2^{(1)}$.

Therefore, if one believes in the consistency of arithmetic (cf. Section 6.6), one has to accept the fact that the sentence Axiom $2^{(1)}$ is not a theorem of this discipline. And from this it follows that Axiom $2^{(1)}$ is not derivable from the remaining axioms of System $\mathfrak{A}^{(1)}$; for otherwise this could not fail to be valid in the case of any interpretation in which all of the remaining axioms hold (cf. analogous considerations in Section 6.2). □

In general, the method of proof by interpretation can be described as follows. It is a question of showing that some sentence A is not a consequence of a certain axiomatic system $\mathfrak{S}$, or other statements of a given deductive theory. For this purpose, we consider an arbitrary deductive theory $\mathfrak{T}$ of which we assume that it is consistent (it may, in particular, be the same theory to which the statements of the axiomatic system $\mathfrak{S}$ belong). We then try to find, within this theory, an interpretation of the system $\mathfrak{S}$ of such a kind that the negation of sentence A is true. If we are successful in doing so, we may apply the law of deduction stated in Section 6.3. As we know, it follows from this law that, if the sentence A could be derived from the statements of the system $\mathfrak{S}$, it would remain valid for any interpretation of this system. Consequently, the very fact of the existence of an interpretation of $\mathfrak{S}$ for which A is not valid is proof that this sentence cannot be derived from system $\mathfrak{S}$. More strictly speaking, it is a proof of the conditional sentence:

If the Theory $\mathfrak{T}$ is consistent, then the sentence A cannot be derived from the statements of the system $\mathfrak{S}$.

The reason why we must include the hypothesis that the theory $\mathfrak{T}$ is consistent is easily seen. For otherwise the theory $\mathfrak{T}$ could contain two contradictory sentences among its axioms and theorems, and we could not conclude that $\mathfrak{T}$ did not contain the sentence A (or rather the interpretation of A), from the mere fact that $\mathfrak{T}$ did contain the negation of A; thus our argument would no longer be valid.

In order to arrive, in the above way, at an exhaustive proof of the independence of a given axiomatic system, the method described has to be applied as many times as there are axioms in the system in question; each axiom in turn is taken as the sentence A, while $\mathfrak{S}$ consists of the remaining axioms of the system.

10.3 Elimination of Superfluous Primitive Terms

We return once more to the axiomatic system $\mathfrak{A}^{(1)}$. Since this system is independent, it does not permit any further simplification by the omission of superfluous axioms. Nevertheless, a simplification can be achieved in a different way. For it turns out that the primitive terms of System $\mathfrak{A}^{(1)}$ are not mutually independent. In fact, either one of the two symbols "$<$" and "$>$" can be stricken from the list of primitive terms, and can then be defined in terms of the other. This is easily seen from Theorem 3; on account of its form, this theorem may be considered as a definition of the symbol "$>$" by means of the symbol "$<$", and if in this theorem we exchange the two sides of the equivalence, we may look upon it as a definition of the symbol "$<$" by means of the symbol "$>$". From the didactical point of view, this reduction of the primitive terms might provoke certain objections; for the terms "$<$" and "$>$" are equally clear in their meaning and the relations denoted by them possess entirely analogous properties, so that it would appear slightly artificial to consider one of these terms immediately comprehensible while the other has first to be defined with the help of other. But these objections carry little conviction.

If now, in disregard of any didactical considerations, we resolve to eliminate one of the symbols in question from the list of primitive terms, the task arises of giving our axiomatic system a form for which no defined terms occur in it. (This is a methodological postulate, by the way, which in practice is frequently disregarded; in geometry, especially, the axioms are usually formulated with the help of defined terms in order to enhance their simplicity and evidence). This task does not present any difficulties; we simply replace in the axiomatic system $\mathfrak{A}^{(1)}$ every formula of the type $x > y$ with $y < x$, which, by Theorem 3, is equivalent to it. It is then easily seen that Axiom 1 may be replaced by the Law of Connectivity (Theorem 4), since each follows from the other on the basis of general laws of logic (of sentential calculus, that is); Axiom 3 now becomes a simple substitution of Axiom 2, and may hence be omitted altogether. In this way, we arrive at the system consisting of the following seven axioms:

Axiom $1^{(2)}$**.** $x \neq y \rightarrow (x < y \vee y < x)$

Axiom $2^{(2)}$**.** $x < y \rightarrow y \not< x$

Axiom $3^{(2)}$**.** $(x < y \wedge y < z) \rightarrow x < z$

Axiom $4^{(2)}$**.** $x + y = y + x$

Axiom $5^{(2)}$**.** $x + (y + z) = (x + y) + z$

Axiom $6^{(2)}$**.** $\forall x, y \; \exists z \; (x = y + z)$

Axiom $7^{(2)}$**.** $y < z \rightarrow x + y < x + z$

This axiomatic system, creatively labelled System $\mathfrak{U}^{(2)}$, is thus equipollent to both Systems $\mathfrak{U}$ and $\mathfrak{U}^{(1)}$. However, in saying this, we commit one inexactitude; for it is impossible to derive from the axioms of System $\mathfrak{U}^{(2)}$ those sentence of System $\mathfrak{U}$ or $\mathfrak{U}^{(1)}$ which contain the symbol "$>$", unless System $\mathfrak{U}^{(2)}$ is enlarged by adding the definition of this symbol. We may, as we know, give this definition the following form:

Definition $1^{(2)}$. $x > y \overset{def}{\longleftrightarrow} y < x.$

We also know that this last sentence can be proved on the basis of Systems $\mathfrak{U}$ or $\mathfrak{U}^{(1)}$, if it is treated, not as a definition, but as an ordinary theorem. The fact of the equipollance of the three systems in question can now be formulated as follows:

Theorem. *System $\mathfrak{U}^{(2)}$ together with Definition $1^{(2)}$ is equipollent to each of Systems $\mathfrak{U}$ and $\mathfrak{U}^{(1)}$.*

An equally cautious mode of formulation is indicated whenever two axiomatic systems are compared which, though equipollent, contain, at least partly, different primitive terms.

The axiomatic system $\mathfrak{U}^{(2)}$ is distinguished advantageously by the simplicity of its structure. The first three axioms concern the relation $<$, and together they assert that the set $\mathbb{R}$ is ordered by this relation; the next three axioms are concerned with addition, and they assert that the set $\mathbb{R}$ is an Abelian group with respect to addition; the last axiom finally, the *Law of Monotony*, states a certain dependence between the relation $<$ and the operation $+$.

Definition 10.1. A class K is said to be an *ordered Abelian group with respect to the relation R and the operation O* if:
(i) the class K is ordered by the relation R,
(ii) the class K is an Abelian group with respect to the operation O,
(iii) the operation is monotonic in K with respect to the relation R.

In accordance with this terminology we can say that the set $\mathbb{R}$ is characterized by the axiomatic system $\mathfrak{U}^{(2)}$ as an ordered Abelian group with respect to the relation $<$ and the operation $+$. The following facts concerning System $\mathfrak{U}^{(2)}$ can be established:

Theorem. *System $\mathfrak{U}^{(2)}$ is an independent axiomatic system, and moreover, all its primitive terms, namely "$\mathbb{R}$", "$<$", and "$+$", are mutually independent.*

We omit the proof of this statement. We remark only that, in order to establish the mutual independence of the primitive terms, one has again to apply the method of proof by interpretation, which in this case assumes a more involved form; lack of space prevents us from going into modifications of that method which would be required for this purpose.

10.4 Further Simplifications of the Axiomatic System

System $\mathfrak{U}^{(2)}$ can obviously be replaced by any system of sentences equipollent to it. We will give here a particulary simple example of such a system, which may be called System $\mathfrak{U}^{(3)}$, and which contains the same primitive terms as $\mathfrak{U}^{(2)}$. It consists of only five sentences:

Axiom $1^{(3)}$. $x \neq y \to (x < y \vee y < x)$

Axiom $2^{(3)}$. $x < y \to y \nless x$

Axiom $3^{(3)}$. $x + (y + z) = (x + z) + y$

Axiom $4^{(3)}$. $\forall x, y\ \exists z\ (x = y + z)$

Axiom $5^{(3)}$. $x + z < y + t \to (x < y \vee z < t)$

Of course, we will have to show that System $\mathfrak{U}^{(3)}$ is equipollent to System $\mathfrak{U}^{(2)}$, leading up to this we will prove a few select axioms from System $\mathfrak{U}^{(2)}$ can be derived from axioms of System $\mathfrak{U}^{(3)}$.

Theorem I′. *Axiom $4^{(2)}$ can be derived from the axioms of System $\mathfrak{U}^{(3)}$.*

Proof.

(1)	$y = x + z$	Instance of Axiom $4^{(3)}$, $x : y$, $y : x$
(2)	$x + (x + z) = (x + z) + x$	Instance of Axiom $3^{(3)}$, $y : x$
(3)	$x + y = y + x$	Rule of Replacement (1) into (2)

□

Theorem II′. *Axiom $5^{(2)}$ can be derived from the axioms of System $\mathfrak{U}^{(3)}$.*

Proof. Since we have already shown Theorem I′ to be true, we are now free to use Axiom $4^{(2)}$ in the derivation of Axiom $5^{(2)}$.

(1)	$x + (z + y) = (x + y) + z$	Instance of Axiom $3^{(3)}$, $y : z$, $z : y$
(2)	$z + y = y + z$	Instance of Axiom $4^{(2)}$, $x : y$, $y : z$
(3)	$x + (y + z) = (x + y) + z$	Rule of Replacement (2) into (1)

□

In order to facilitate the derivation of Axioms $3^{(2)}$ and $7^{(2)}$, we shall first show how some of the axioms and theorems stated in the preceding chapters may be proved on the basis of System $\mathfrak{U}^{(3)}$.

Theorem III′. *Theorem 1 can be derived from the axioms of System* $\mathfrak{U}^{(3)}$.

Proof. We merely observe that the proof of Theorem 1 given in Section 8.2 is based exclusively upon Axiom 2, which in turn coincides with Axiom $2^{(3)}$ of system $\mathfrak{U}^{(3)}$. □

Theorem IV′. *Axiom 6 can be derived from the axioms of System* $\mathfrak{U}^{(3)}$.

Proof. In fact, we saw in Section 10.1 that Axiom 6 can be deduced from Axioms 7, 8, and 9. Axioms 7 and 8 are the same as axioms $4^{(2)}$ and $5^{(2)}$, and can therefore, by Theorems I′ and II′, be derived from the axioms of System $\mathfrak{U}^{(3)}$. Axiom 9, on the other hand, occurs as Axiom $4^{(3)}$ in System $\mathfrak{U}^{(3)}$. Hence, Axiom 6 is a consequence of the axioms of System $\mathfrak{U}^{(3)}$. □

Theorem V′. *Theorem 11 can be derived from the axioms of System* $\mathfrak{U}^{(3)}$.

Proof. In the second proof of Theorem 11, as given in Section 9.3, only Axioms 7, 8, and 9 were used. Theorem 11 is therefore derivable from the axioms of System $\mathfrak{U}^{(3)}$ for the same reason as Axiom 6 (see the previous theorem). □

Theorem VI′. *Theorem 12 can be derived from the axioms of System* $\mathfrak{U}^{(3)}$.

Proof.

(1)	$x + y < x + z$	Assume hypothesis
(2)	$x + y < x + z \rightarrow (x < x \vee y < z)$	Instance of Axiom $5^{(3)}$, $y : x,\ z : y,\ t : z$
(3)	$x < x \vee y < z$	Rule of Detachment, (1) and (2)
(4)	$x \nless x \rightarrow y < z$	Instance of $(\sim p \vee q) \leftrightarrow (p \rightarrow q)$ $p :\sim x < x,\ q : y < z$
(5)	$x \nless x$	Theorem 1
(6)	$y < z$	Rule of Detachment, (4) and (5)

□

Theorem VII′. *Axiom* $3^{(2)}$ *can be derived from the axioms of System* $\mathfrak{U}^{(3)}$.

Proof.

(1)	$x < y \wedge y < z$	Assume hypothesis
(2)	$x < y \wedge y < z \rightarrow x < y$	Law of And, Breaking Apart
(3)	$x < y \wedge y < z \rightarrow y < z$	Law of And, Breaking Apart

(4)	$x < y$	Rule of Detachment, (1) and (2)
(5)	$y < z$	Rule of Detachment, (1) and (3)
(6)	$y + x = y + z \to x = z$	Instance of Theorem 11, $x : y$, $y : x$
(7)	$y + x = y + z$	Secondary assumption
(8)	$x = z$	Rule of Detachment, (6) and (7)
(9)	$z < y$	Rule of Replacement (8) into (4)
(10)	$z < y \to y \not< z$	Instance of Axiom $2^{(3)}$, $x : z$
(11)	$y \not< z$	Rule of Detachment, (9) and (10)
(12)	$y + x = y + z \to y \not< z$	Instance of $\text{True}_1 \to \text{True}_2$ $\text{True}_1 : y + x = y + z$, $\text{True}_2 : y \not< z$
(13)	$(y + x = y + z \to y \not< z) \wedge y < z$	Rule of And, Joining Together on (12) and (5)
(14)	$[\,(y + x = y + z \to y \not< z) \wedge y < z\,] \to y + x \neq y + z$	Law of General Contradiction
(15)	$y + x \neq y + z$	Rule of Detachment, (13) and (14)
(16)	$y + x \neq y + z \to y + x < y + z \vee y + z < y + x$	Instance of Axiom $1^{(3)}$, $x : y + x$, $y : y + z$
(17)	$y + x < y + z \vee y + z < y + x$	Rule of Detachment, (15) and (16)
(18)	$(y + x < y + z \vee y + z < y + x) \leftrightarrow y + x \not< y + z \to y + z < y + x$	Instance of $(p \vee q) \leftrightarrow (\sim p \to q)$ $p : y + x \not< y + z$, $q : y + z < y + x$
(19)	$y + x \not< y + z \to y + z < y + x$	Substitute (18) into (17)
(20)	$y + x \not< y + z$	Secondary assumption
(21)	$y + z < y + x$	Rule of Detachment, (19) and (20)
(22)	$x + y = y + x$	Axiom $4^{(2)}$
(23)	$y + z < x + y$	Rule of Replacement (22) into (21)

(24)	$y + z < x + y \to y < x \vee z < y$	Instance of Axiom $5^{(3)}$ $x : y,\ y : x,\ t : y$
(25)	$y < x \vee z < y$	Rule of Detachment, (23) and (24)
(26)	$x < y \to y \not< x$	Axiom $2^{(3)}$
(27)	$y \not< x$	Rule of Detachment, (26) and (4)
(28)	$(y < x \vee z < y) \leftrightarrow$ $y \not< x \to z < y$	Instance of $(p \vee q) \leftrightarrow (\sim p \to q)$ $p : y < x,\ q : z < y$
(29)	$y \not< x \to z < y$	Substitute (28) into (25)
(30)	$z < y$	Rule of Detachment, (29) and (27)
(31)	$y < z \to z \not< y$	Instance of Axiom $2^{(3)}$ $x : y,\ y : z$
(32)	$z \not< y$	Rule of Detachment, (5) and (31)
(33)	$y \not< x \wedge z \not< y$	Rule of And, Joining Together on (27) and (32)
(34)	$y \not< x \wedge z \not< y$ $\leftrightarrow \sim (y < x \vee z < y)$	De Morgan's Law on (33)
(35)	$\sim (y < x \vee z < y)$	Substitute (34) into (33)
(36)	$y + x \not< y + z \to$ $\sim (y < x \vee z < y)$	Instance of $\text{True}_1 \to \text{True}_2$ $\text{True}_1 : y + x \not< y + z,$ $\text{True}_2 : (y < x \vee z < y)$
(37)	$[\,y + x \not< y + z \to$ $\sim (y < x \vee z < y)\,]$ $\wedge\, (y < x \vee z < y)$	Rule of And, Joining Together on (36) and (25)
(38)	$[\,[\,y + x \not< y + z \to$ $\sim (y < x \vee z < y)\,]$ $\wedge\, (y < x \vee z < y)\,]$ $\to (y + x < y + z)$	Law of General Contradiction
(39)	$y + x < y + z$	Rule of Detachment, (37) and (38)
(40)	$y + x < y + z \to x < z$	Instance of Theorem 12,

$x : y,\ y : x$

(41) $x < z$ Rule of Detachment, (39) and (40)

This proof required the use of Theorems 11 and 12 which were made available by proving Theorems V′ and VI′. Furthermore, we had to prove this by contradiction and also by exhausting all possible scenarios. By Axiom $1^{(3)}$, if $y + x \neq y + z$, then either $y + x < y + z$ or $y + z < y + x$. So first, we assumed $y + x = y + z$ and arrived at a contradiction in statement (11), then used the rule of detachment on this result to consider the two remaining cases: $y + x < y + z$ or $y + z < y + x$. Of course, we wanted to eventually arrive at $y + x < y + z$, so by assuming to the contrary that $y + x \not< y + z$, we must have $y + z < y + x$. Under this assumption, we yet again arrive at a contradiction in statement (35). Hence, it must be true that $y + x < y + z$, in which case, the conclusion of Axiom $3^{(2)}$ is derived in sentences (4) and (41) with the assistance of Theorem 12. □

Theorem VIII′. *Axiom $7^{(2)}$ can be derived from the axioms of System $\mathfrak{A}^{(3)}$.*

Proof.

(1) $y < z$ Assume hypothesis

(2) $x + y = x + z \to y = z$ Theorem 11,

(3) $x + y = x + z$ Secondary assumption

(4) $y = z$ Rule of Detachment, (2) and (3)

(5) $z < z$ Rule of Replacement (4) into (1)

(6) $z \not< z$ Instance of Theorem 1, $x : z$

(7) $x + y = x + z \to z < z$ Instance of $\text{True}_1 \to \text{True}_2$
$\text{True}_1 : x + y = x + z$,
$\text{True}_2 : z < z$

(8) $(x + y = x + z \to z < z)$
$\wedge\, z \not< z$ Rule of And, Joining Together on (7) and (6)

(9) $[\,(x + y = x + z \to z < z)$
$\wedge\, z \not< z\,]$
$\to x + y \neq x + z$ Law of General Contradiction

(10) $x + y \neq x + z$ Rule of Detachment, (8) and (9)

(11) $x + y \neq x + z \to$ Instance of Axiom $1^{(3)}$,

	$x + y < x + z \vee x + z < x + y$	$x : x + y,\ y : x + z$
(12)	$x + y < x + z \vee x + z < x + y$	Rule of Detachment, (10) and (11)
(13)	$(x + y < x + z \vee x + z < x + y) \leftrightarrow$ $x + y \not< x + z \rightarrow x + y < x + z$	Instance of $(p \vee q) \leftrightarrow (\sim p \rightarrow q)$ $p :,\ q : x + z < x + y$
(14)	$x + y \not< x + z \rightarrow x + y < x + z$	Substitute (13) into (12)
(15)	$x + y \not< x + z$	Secondary assumption
(16)	$x + y < x + z$	Rule of Detachment, (14) and (15)
(17)	$x + y < x + z \rightarrow z < y$	Instance of Theorem 12, $y : z,\ z : y$
(18)	$z < y$	Rule of Detachment, (16) and (17)
(19)	$z < y \rightarrow y \not< z$	Instance of Axiom $2^{(3)}$,$x : z$
(20)	$y \not< z$	Rule of Detachment, (18) and (19)
(21)	$x + y \not< x + z \rightarrow y \not< z$	Instance of $\mathrm{True}_1 \rightarrow \mathrm{True}_2$ $\mathrm{True}_1 : x + y \not< x + z$, $\mathrm{True}_2 : y \not< z$
(22)	$(x + y \not< x + z \rightarrow y \not< z)$ $\wedge\, y < z$	Rule of And, Joining Together on (21) and (1)
(23)	$[\,(x + y \not< x + z \rightarrow y \not< z)$ $\wedge\, y < z\,]$ $\rightarrow x + y < x + z$	Law of General Contradiction
(24)	$x + y < x + z$	Rule of Detachment, (22) and (23)

The proof here was very similar to that of the previous theorem with regard to the fact that we had to exhaust all three possibilities of relations between the two numbers $x + y$ and $x + z$. □

We have seen in this manner that all sentences of System $\mathfrak{U}^{(2)}$ are consequences of System $\mathfrak{U}^{(3)}$, and conversely; thus the two axiomatic systems $\mathfrak{U}^{(3)}$ and $\mathfrak{U}^{(3)}$ are equipollent.

System $\mathfrak{U}^{(3)}$, no doubt, is simpler than System $\mathfrak{U}^{(2)}$, and hence still simpler than Systems $\mathfrak{U}$ or $\mathfrak{U}^{(1)}$. Particularly interesting is a comparison between Systems $\mathfrak{U}$ or $\mathfrak{U}^{(3)}$; as a result of the successive reductions that have been carried out, the original number of axioms has been diminished by more than one half. On the other hand, it should be noted that some of the sentences of System $\mathfrak{U}^{(3)}$ (namely Axioms $3^{(3)}$ and $5^{(3)}$) are less natural and simple than the axioms of the other systems, and also that the proofs of some, even very elementary, theorems are here comparatively more difficult and involved than on the basis of those other systems.

Just like a system of axioms, a system of primitive terms may be replaced by any equipollent system. This applies, in particular, to the system of three terms "$\mathbb{R}$", "$<$", and "$+$" which occur as the only primitive terms in the axioms last considered. If, for instance, in this system we replace the symbol "$<$" by "$\leq$", we obtain an equipollent system; for the second of these symbols was defined in terms of the first, and Theorem 8 tells us how the first may be defined by means of the second. But such a transformation of the system of primitive terms would be in no way advantageous; in particular, it would contribute nothing to a simplification of the axioms, and to the reader, who is possible more familiar with the symbol "$<$" than with the symbol "$\leq$", it might even appear rather artificial. Another equipollent system can be obtained by replacing in the original system the symbol "$+$" by "$-$"; but, again, this transformation would not be at all expedient. In conclusion we should note that other systems of primitive terms are known which are equipollent to the system in question and consist of but two terms.

10.5 Problem of the Consistency of the Constructed Theory

We shall now briefly touch on some other methodological problems concerning the fragment of arithmetic considered above; these are the problems of consistency and of completeness (cf. Section 6.6). Since it is quite irrelevant whether we refer our remarks to one or another of several equipollent systems, we shall now always speak of System $\mathfrak{U}$.

If we believe in the consistency of the whole of arithmetic (and this assumption has been made previously and will be made again in our further considerations), then we must all the more accept the fact that

The mathematical theory based on System $\mathfrak{U}$ is consistent.

But while the attempts to give a strict proof of the consistency of the whole of arithmetic have met with essential difficulties (cf. Section 6.6), a proof of this kind for System $\mathfrak{U}$ is not only possible but even comparatively simple. One reason for this is the fact that the variety of theorems which can be derived from System $\mathfrak{U}$ is very small indeed; it is, for instance, not possible to give, on its basis, an answer to the very elementary question as to whether any numbers exist at all. This circumstance facilitates considerably the proof of the fact that the part of arithmetic considered does not contain a single pair of contradictory theorems. With the means here at our disposal, however, it would be a hopeless undertaking to sketch the proof of the consistency or even to acquaint the reader with its fundamental idea; this would require a much deeper knowledge of logic, and an essential preliminary task would be the reconstruction of the part of arithmetic in question as a formalized deductive theory (cf. Section 6.5). It may be added that if System $\mathfrak{U}$ is enriched by a single sentence to the effect that at least two distinct numbers exist, then the attempt to prove the consistency of this particular axiomatic system thus extended will meet with difficulties of some degree as are encountered in the case of the entire system of arithmetic.

10.6 Problem of the Completeness of the Constructed Theory

In comparison with the question of consistency, that of the completeness of System $\mathfrak{U}$ can be dealt with much more readily.

There are numerous problems, formulated exclusively in logical terms of System $\mathfrak{U}$, that do not in any way admit of a decision based on this system. One such problem has already been mentioned in the preceding section. Another example is given by the sentence:

$$\forall x \; \exists y \; (x = y + y) \tag{10.1}$$

On the basis of System $\mathfrak{U}$ alone, it is impossible either to prove or disprove this sentence. That this is so can be seen from the following consideration. By replacing the symbol "$\mathbb{R}$", which stands for the set of all real numbers, with the set "$\mathbb{Q}$" or "$\mathbb{Z}$", which, respectively, stand for the sets of all rational numbers and all integers, all axioms of our system still hold. For the set "$\mathbb{Q}$", sentence (10.1) still holds, since any rational number divided by two is another rational number. However, for the set "$\mathbb{Z}$", sentence (10.1) is false. Only even integers x, when divided by two, yield another integer y. If, therefore, we succeeded in proving this sentence on the basis of System $\mathfrak{U}$, we would arrive at a contradiction within the arithmetic of integers; if, on the other hand, we were able to disprove it, we would find ourselves involved in a contradiction within the arithmetic of rational numbers.

The argument sketched just now falls under the category of proofs by interpretation (cf. Sections 6.2 and 10.2); in order to make this clear let us reformulate the argument slightly. System $\mathfrak{U}$ has as its primitive terms $\{\mathbb{R}, <, >, +\}$. In the two interpretations above, only the first primitive term was changed, and the primitive terms are given by the sets $\{\mathbb{Q}, <, >, +\}$ and $\{\mathbb{Z}, <, >, +\}$. All axioms of System $\mathfrak{U}$ retain their validity in both interpretations. Sentence (10.1), however, is fulfilled only in the case of the first interpretation, while in the case of the second interpretation, the negation of sentence (10.1) holds.

On the assumption of the consistency of arithmetic we conclude from the first interpretation that the sentence in question cannot be disproved on the basis of System $\mathfrak{U}$, and from the second interpretation we conclude that it also cannot be proved.

We have thus shown that there exist two contradictory sentences, formulated exclusively in logical terms and in primitive terms of mathematical theory which we have been considering, with the property that neither of them can be derived from the axioms of that theory. Consequently, we have:

The mathematical theory based on System $\mathfrak{U}$ is incomplete.

Exercises

1. Let us agree that the symbol "$\otimes$" is defined on the set $\mathbb{R}$ as follows:

$$x \otimes y \overset{def}{\longleftrightarrow} x + 1 < y.$$

Now replace in the axioms of System $\mathfrak{U}^{(2)}$ of Section 10.3, the symbol "$<$" by "$\otimes$" and determine which of the axioms retain their validity and which do not, and hence infer that Axiom $1^{(2)}$ cannot be derived from the remaining axioms.

2. Following the lines of the independence proof sketched in Section 10.2 for Axiom $2^{(1)}$, show that Axiom $2^{(2)}$ cannot be derived from the remaining axioms of $\mathfrak{U}^{(2)}$.

3. Define the set $\mathbb{Z}_3 = \{0, 1, 2\}$, and the relation $\overset{3}{<}$ as holding only in the following three cases:

$$0 \overset{3}{<} 1, \quad 1 \overset{3}{<} 2, \quad 2 \overset{3}{<} 0$$

Further, we define the operation $\overset{3}{+}$ on the elements of $\mathbb{Z}_3$ by the following formulas:

$$0 \overset{3}{+} 0 = 1 \overset{3}{+} 2 = 2 \overset{3}{+} 1 = 0$$
$$0 \overset{3}{+} 1 = 1 \overset{3}{+} 0 = 2 \overset{3}{+} 2 = 1$$
$$0 \overset{3}{+} 2 = 1 \overset{3}{+} 1 = 2 \overset{3}{+} 0 = 2$$

Now, replace, in the axioms of System $\mathfrak{U}^{(2)}$, the primitive terms $\{\mathbb{R}, <, +\}$ by the set of primitive terms $\{\mathbb{Z}_3, \overset{3}{<}, \overset{3}{+}\}$. Show that Axiom $3^{(2)}$ cannot be derived from the remaining axioms with this interpretation.

4. In order to show by means of a proof by interpretation that Axiom $4^{(2)}$ is not derivable from the remaining axioms of System $\mathfrak{U}^{(2)}$, it is sufficient to replace the symbol "+" by the symbol of a certain one among the four operations mentioned in Exercise 2 of Chapter 9. Which is the operation that has to be used? Be sure to explain your reasoning.

5. Consider the operation $\oplus$ satisfying the following formula:

$$x \oplus y = 2 \cdot (x + y).$$

Show, with the help of this operation, that Axiom $5^{(2)}$ cannot be deduced from the other axioms of System $\mathfrak{U}^{(2)}$.

6. Construct a set of numbers such that, together with the relation $<$ and the operation $+$, it fails to satisfy Axiom $6^{(2)}$ but forms a model of the remaining axioms of System $\mathfrak{U}^{(2)}$. What conclusion may, therefore, be drawn with respect to the possibility of deriving Axiom $6^{(2)}$?

7. In order to show that Axiom $7^{(2)}$ is not a consequence of all the other axioms of System $\mathfrak{U}^{(2)}$, one can proceed by replacing in all axioms two of the primitive terms of the system by corresponding symbols introduced in Exercise 3, leaving the third primitive term unchanged. Determine which term should be left unchanged. Once again, explain your reasoning.

8. The results obtained in Exercises 1–7 go to show that none of the axioms of System $\mathfrak{U}^{(2)}$ can be derived from the remaining axioms of that system. Carry out an analogous proof of independence for System $\mathfrak{U}^{(1)}$, using in part, the interpretations applied in the previous exercises.

9. Repeat Exercise 8, except for System $\mathfrak{U}^{(3)}$.

10. Show, on the basis of System $\mathfrak{U}^{(2)}$, that any set of numbers which is an Abelian group with respect to $+$ is at the same time an ordered Abelian group with respect to the relation $<$ and the operation of $+$.

11. In Exercise 4 of Chapter 9 several sets of numbers were given which form Abelian groups with respect to $\cdot$ (the operation of multiplication). Which of these sets are ordered Abelian groups with respect to the relation $<$ and the operation of $\cdot$?

12. Show that the system of Axioms $1^{(2)}$–$3^{(2)}$ is equipollent to the system consisting of Axiom $1^{(1)}$ and the following sentence:

Axiom. $(x < y \wedge y < z \wedge z < t \wedge t < u \wedge u < v) \rightarrow v \nless x$

13. As a generalization of Exercise 12, establish the following general law of the theory of relations:

Theorem. *For the class K to be ordered by the relation R it is necessary and sufficient that R is connected in K and that it satisfies the following condition:*

$$(xRy \wedge yRz \wedge zRt \wedge tRu \wedge uRv) \rightarrow \sim vRx.$$

14. Using the considerations of Sections 9.2, 10.1, and 10.4, show that the following three systems of sentences are equipollent:
(a) The system of Axioms 6–9
(b) The system of Axioms $4^{(2)}$–$6^{(2)}$
(c) The system of Axioms $3^{(3)}$ and $4^{(3)}$

15. Generalizing the result of Exercise 14, formulate new definitions of the class K being an Abelian group with respect to the operation O, that are equivalent, but simpler than, the definition given in Section 9.1.

16. Consider the axiomatic system $\mathfrak{A}^{(4)}$ consisting of the following five axioms:

Axiom $1^{(4)}$. $x \neq y \rightarrow (x < y \vee y < x)$

Axiom $2^{(4)}$. $(x < y \wedge y < z \wedge z < t \wedge t < u \wedge u < v) \rightarrow v \not< x$

Axiom $3^{(4)}$. $x + (y + z) = (x + z) + y$

Axiom $4^{(4)}$. $\forall x, y\ \exists z\ (x = y + z)$

Axiom $5^{(4)}$. $y < z \rightarrow x + y < x + z$

Using the results of Exercises 12 and 14, show that $\mathfrak{A}^{(4)}$ is equipollent to each of the Systems $\mathfrak{A}^{(2)}$ and $\mathfrak{A}^{(3)}$.

17. Consider the axiomatic system $\mathfrak{A}^{(5)}$ consisting of the following seven axioms:

Axiom $1^{(5)}$. $\forall x, y\ (x \leq y \vee y \leq x)$

Axiom $2^{(5)}$. $(x \leq y \wedge y \leq x) \rightarrow x = y$

Axiom $3^{(5)}$. $(x \leq y \wedge y \leq z) \rightarrow x \leq z$

Axiom $4^{(5)}$. $x + y = y + x$

Axiom $5^{(5)}$. $x + (y + z) = (x + y) + z$

Axiom $6^{(5)}$. $\forall x, y\ \exists z\ (x = y + z)$

Axiom $7^{(5)}$. $y \leq z \rightarrow x + y \leq x + z$

Show that Systems $\mathfrak{A}^{(2)}$ and $\mathfrak{A}^{(5)}$ become equipollent if Definition 1 of Section 8.4 is added to the first, and Theorem 8 of Section 8.4 to the second, considering the latter theorem as a definition of the symbol "$<$". Why may we not simply say that $\mathfrak{A}^{(2)}$ and $\mathfrak{A}^{(5)}$ are equipollent?

Chapter 11

Foundations of Arithmetic for Real Numbers

11.1 First Axiomatic System for the Arithmetic of Real Numbers

The axiomatic system $\mathfrak{U}$ is insufficient as a foundation for the whole of the arithmetic of real numbers, because – as has been seen in Section 10.6 – numerous theorems of this discipline cannot be deduced from the axioms of this system, and also for another, no less important and, incidentally, quite analogous reason: a number of concepts belonging to the field of arithmetic can be found that are not definable with the help of the primitive terms occurring in System $\mathfrak{U}$. Thus, System $\mathfrak{U}$ does not enable us to define the symbols of multiplication or division, or even such symbols as "1", "2", and so on.

The question at once presents itself as to how we have to transform and supplement our system of axioms and primitive terms in order to arrive at a sufficient basis for the construction of the entire arithmetic of real numbers. This problem can be solved in a variety of ways. Two different methods of solution will be sketched here.[1]

In the case of the first method, we choose as our point of departure the system $\mathfrak{U}^{(3)}$ (cf. Section 10.4); to the primitive terms appearing in that system we add the word "*one*" which, as usual, will be replaced by the symbol "1", and the axioms of the system are supplemented by four new sentences. In this way, a new system $\mathfrak{U}'$ is obtained, containing the set of four primitive terms $\{\mathbb{R}, <, +, 1\}$ and consisting of the nine axioms which we shall list explicitly below:

Axiom 1′. $x \neq y \rightarrow (x < y \vee y < x)$

Axiom 2′. $x < y \rightarrow y \nless x$

Logic, New New Edition.
Karl Frinkle.

Axiom $3'$**.** $\forall\, x, z\ \exists\, y\, [\, x < z \to\ (x < y \wedge y < z)\,]$

Axiom $4'$**.** *If* $K, L \subseteq \mathbb{R}$ *satisfying the condition:*

$$\forall\, x \in K, y \in L\ (x < y),$$

then the following condition holds:

$$\exists\, z\, [\, (\, x \in K \wedge y \in L \wedge x \neq z \wedge y \neq z\,) \to (\, x < z \wedge z < y\,)\,].$$

Axiom $5'$**.** $x + (y + z) = (x + z) + y$

Axiom $6'$**.** $\forall\, x, y\ \ \exists\, z\ (x = y + z)$

Axiom $7'$**.** $x + z < y + t \to (x < y \vee z < t)$

Axiom $8'$**.** $1 \in \mathbb{R}$

Axiom $9'$**.** $1 < 1 + 1$

11.2 Characterization of the First Axiomatic System

The axioms listed in the preceding section fall into three groups. In the first group, consisting of Axioms $1'$-$4'$, only the top primitive terms "$\mathbb{R}$" and "$<$" occur; in the second group, to which Axioms $5'$-$7'$ belong, we have the additional symbol "$+$"; finally, in the third group, in which we have Axioms $8'$ and $9'$, the new symbol "1" appears.

Among the axioms of the first group there are two which we had not met before, namely Axioms $3'$ and $4'$. Axiom $3'$ is called the *Law of Density* for the relation $<$ – it expresses the fact that this relation is dense in the set of all numbers.

Definition 11.1. A relation R is *dense in the class* K if the following property holds:

$$\forall\, x, y\ \exists\, z\ [\, xRy \to (xRz \wedge zRy)\,]$$

Axiom $4'$ is known as the *Law of Continuity* for the relation $<$, or, also, as *Dedekind's Axiom*[2]; in order to state in general, under what condition the relation R is called continuous, we remove primitives from Axiom $4'$:

Definition 11.2. A relation R is *continuous in the class* K if the following property holds: If $L, M \subseteq K$ satisfying the condition:

$$\forall\, x \in L, y \in M\ \ (x < y),$$

then the following condition holds:

$$\exists\, z\, [\, (\, x \in L \wedge y \in M \wedge x \neq z \wedge y \neq z\,) \to (\, x < z \wedge z < y\,)\,].$$

Definition 11.3. A relation R is *densely ordered in the class* K if the relation is both dense and ordered in the class K.

Definition 11.4. A relation R is *continuously ordered in the class* K if the relation is both continuous and ordered in the class K.

Axiom $4'$ is intuitively less evident and more complicated than the remaining axioms; for one thing, it differs from the other axioms inasmuch as it is concerned, not with individual numbers, but with sets of numbers. In order to give this axiom a simple and more comprehensible form, it is expedient to have it preceded by the following definitions:

Definition 11.5. We say that the set of numbers K *precedes* the set of numbers L if, and only if, every number K is less than every number L.

Definition 11.6. We say that the number z *separates* the sets of numbers K and L if, and only if, for any two elements $x \in K$ and $y \in L$, both disjoint from z, we have: $x < z$ and $z < y$.

Axiom of Continuity. *If one set of numbers precedes another, then there exists at least one number separating the two sets.*

All the axioms of the second group are already known to us from earlier considerations. The axioms of the third group, though new, have no simple and obvious a content that they hardly require any comment. We might only remark this much, that if Axiom $9'$ is preceded by definitions of the symbol "0" and of the expression "*positive number*", then it may be replaced either by the formula:

$$0 < 1$$

or else by the sentence:

1 is a positive number.

The Axioms $1'$, $2'$, $5'$, $6'$, and $7'$ form just what we called System $\mathfrak{U}^{(2)}$ – characterizes the set of all numbers as an ordered Abelian group (cf. Section 10.1). Considering the content of the newly added Axioms $3'$, $4'$, $8'$, and $9'$, we may now describe the whole system as follows:

System $\mathfrak{U}'$ expresses the fact that the set of all numbers is a densely and continuously ordered Abelian group with respect to the relation $<$ and the operation $+$, and it singles out a certain positive element 1 in that set.

From the methodological point of view, System $\mathfrak{U}'$ possesses several advantages. Formally considered, it appears to be the simplest of all known axiomatic systems that form a sufficient basis upon which to found the entire system of arithmetic. With the exception of Axiom $1'$, which – though not quite easily –

can be derived from the remaining axioms, all the other axioms of the system as well as the primitive terms occurring in these axioms are mutually independent. The didactical value of System $\mathfrak{A}'$, on the other hand, is far smaller, because the simplicity of the foundations causes considerable complications in the further construction. Even the definition of multiplication and the derivation of the basic laws for this operation are not easy to carry through. Almost from the very beginning, the arguments will have to make essential use of the Axiom of Continuity. Without its help, for instance, it would not be possible to prove, on the basis of System $\mathfrak{A}'$, the existence of the number $\frac{1}{2}$, i.e. of a number y such that $y + y = 1$, and the inferences based on that axiom are usually found rather difficult by the casual reader.

11.3 Second Axiomatic System for the Arithmetic of Real Numbers

For the reasons mentioned above it is worth while to search for a different axiomatic system upon which to construct arithmetic. A system of this kind can be obtained in the following way. As our point of departure we use System $\mathfrak{A}^{(2)}$. Three new primitive terms will be adopted: "*zero*", "*one*", and "*product*"; the first two will, as usual, be replaced by the symbols "0" and "1", while multiplication (or product of two numbers) of x and y will be denoted by "$x \cdot y$". Further, thirteen new axioms will be added; of these, two are already known to us, namely the Axiom of Continuity and the Law of Performability for addition. We thus finally arrive at $\mathfrak{A}''$ containing the set six primitive terms $\{\mathbb{R}, <, +, 1, \cdot, 1\}$, and consisting of the following twenty sentences:

Axiom $1''$**.** $x \neq y \rightarrow (x < y \vee y < x)$

Axiom $2''$**.** $x < y \rightarrow y \not< x$

Axiom $3''$**.** $(x < y \wedge y < z) \rightarrow x < z$

Axiom $4''$**.** *If $K, L \subseteq \mathbb{R}$ satisfying the condition:*

$$\forall x \in K, y \in L \ (x < y),$$

then the following condition holds:

$$\exists z [(x \in K \wedge y \in L \wedge x \neq z \wedge y \neq z) \rightarrow (x < z \wedge z < y)].$$

Axiom $5''$**.** $\forall y, z \ \exists x \ (x = y + z)$

Axiom $6''$**.** $x + y = y + x$

Axiom $7''$**.** $x + (y + z) = (x + y) + z$

Axiom $8''$**.** $\forall\, x, y\ \exists\, z\ \ (x = y + z)$

Axiom $9''$**.** $y < z \rightarrow x + y < x + z$

Axiom $10''$**.** $0 \in \mathbb{R}$

Axiom $11''$**.** $x + 0 = x$

Axiom $12''$**.** $\forall\, x, y\ \exists\, z\ \ (x = y \cdot z)$

Axiom $13''$**.** $x \cdot y = y \cdot x$

Axiom $14''$**.** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Axiom $15''$**.** $\forall\, x, y\ \exists z\ (y \neq 0 \rightarrow x = y \cdot z)$

Axiom $16''$**.** $(\,0 < x \wedge y < z\,) \rightarrow x \cdot y < x \cdot z$

Axiom $17''$**.** $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Axiom $18''$**.** $1 \in \mathbb{R}$

Axiom $19''$**.** $x \cdot 1 = x$

Axiom $20''$**.** $0 \neq 1$

11.4 Characterization of the Second Axiomatic System

In System $\mathfrak{A}''$, as in System $\mathfrak{A}'$, three groups of axioms may be distinguished. In Axioms $1''$-$4''$, which form the first group, we have only the two primitive terms "$\mathbb{R}$" and "$<$"; the second group, consisting of Axioms $5''$-$11''$, contains the two further symbols of "+" and "0"; finally, the third group, which is made up of Axioms $12''$-$20''$, involves primarily "$\cdot$" and "1".

All axioms of the first two groups, with the exception of Axioms $10''$ and $11''$, are already known to us. Axioms $10''$ and $11''$ together state that 0 is a (right-hand) unit element of the operation of addition.

Definition 11.7. The element u is said to be a *right-handed unit element of the operation* O, if $u \in K$ and:

$$x \in K \rightarrow x\, O\, u = x.$$

Definition 11.8. The element u is said to ba a *left-handed unit element of the operation* O, if $u \in K$ and:

$$x \in K \rightarrow u\, O\, x = x.$$

Definition 11.9. The element u is said to ba a *unit element of the operation* O, if u is both a left- and right-handed unit element.

Evidently, in the case of a commutative operation O, every right- or left-handed unit element is simply a unit element.

In the first three axioms of the third group, i.e. Axioms $12''$-$14''$, we recognize the Law of Performability and the Commutative and Associative laws for multiplication; they correspond precisely to Axioms $5''$-$7''$. Axioms $15''$ and $16''$ are called the *Law of Right Invertibility* for multiplication and the *Law of Monotony* for multiplication with respect to the relation $<$. These axioms correspond to the laws of invertibility and monotony for addition, but not quite exactly. The difference lies in the fact that their hypotheses contain the restrictive conditions "$y \neq 0$" and "$0 < x$"; in spite of their names, therefore, they do not permit us to assert simply that multiplication is invertible, or that it is monotonic with respect to the relation $<$ (in the sense of Sections 9.1 and 9.3).

Axiom $17''$ establishes a fundamental connection between addition and multiplication; it is the *Distributive Law* (or strictly speaking, the *Law of Right Distributivity*) for multiplication with respect to addition.

Definition 11.10. The operation P is called *right-distributive with respect to the operation* O *in the class* K if:

$$\forall\, x, y, z\ (\, x\, P\, (y\, O\, z) = (x\, P\, y)\, O\, (x\, P\, z)\,).$$

Definition 11.11. The operation P is called *left-distributive with respect to the operation* O *in the class* K if:

$$\forall\, x, y, z\ (\, (x\, O\ y)\, P\, z = (x\, P\, z)\, O\, (y\, P\, z)\,).$$

If the operation P is commutative, the notions of right and left distributivity coincide, and we simply say that the operation P is *distributive with respect to the operation* O *in the class* K.

The last three axioms concern the number 1. Axioms $18''$ and $19''$ together state that 1 is a right-hand unit element of the operation of multiplication. The content of Axiom $20''$ does not call for any explanation; the role played by this axiom in the construction of arithmetic is greater than might at first be supposed, for without its help it is impossible to show that the set of all numbers is infinite.

In order to describe briefly the totality of properties attributed to addition and multiplication in Axioms $5''$-$8''$, $12''$-$15''$, and $17''$, one says that these axioms characterize $\mathbb{R}$ as a *field* (or, more precisely, a *commutative field*) *with respect to the operations of addition and multiplication.* If, in addition, the axioms of order $1''$-$3''$ and the axioms of monotony $9''$ and $16''$ are taken into account, the set $\mathbb{R}$ is said to be characterized as an *ordered field with respect to the relation* $<$ *and the operations of* $+$ *and* $\cdot$. The reader will easily guess how the concepts of a field and of an ordered field are to be extended to arbitrary classes, operations, and relations. If, finally, the continuity axiom $4''$ and the axioms concerning the numbers 0 and 1, i.e. Axioms $10''$, $11''$, $18''$-$20''$, are taken into consideration, then the content of the entire axiomatic system $\mathfrak{A}''$ may be described as follows:

System $\mathfrak{A}''$ *expresses the fact that the set of all real numbers is a continuously ordered field with respect to the relation* $<$ *and the operations of* $+$ *and* $\cdot$*, and singles out two distinct elements* 0 *and* 1 *in that set, of which the first is the unit element of* $+$ *and the second the unit element of* $\cdot$.

11.5 Equipollence of the Two Systems

The axiomatic systems $\mathfrak{A}'$ and $\mathfrak{A}''$ are equipollent (or rather, they become equipollent as soon as the first system is supplemented by the definitions of the symbol "0" and of the multiplication sign "·", which can be formulated with the help of its primitive terms). However, the proof of this equipollence is not easy. It is true that the derivation of the axioms of the first system from those of the second is not especially difficult; but as far as the opposite task is concerned, it already follows from our earlier remarks that, on the basis of the first system, both the definition of multiplication and the proof of the basic laws governing this operation (which occur as axioms in the second system) present considerable difficulties.

In methodological respects System $\mathfrak{A}'$ surpasses System $\mathfrak{A}''$ considerably. The number of axioms in $\mathfrak{A}''$ is more than twice as large. The axioms are not mutually independent; thus for instance, Axioms $5''$ and $12''$, i.e. the Laws of Performability for addition and multiplication, are derivable from the remaining axioms, or, if these two axioms are retained, certain others such as Axioms $6''$, $11''$, and $14''$ may be eliminated. The primitive terms, too, are not independent, for three of them, namely "<", "0", and "1", can be defined in terms of the others (one of the possible definitions of the symbol "0" has been stated in Section 9.7), and consequently the number of axioms can be further reduced.

We see, therefore, that System $\mathfrak{A}''$ admits of important simplification of various kinds; but as a consequence of these simplifications the didactical advantages of the system would be diminished considerably. And these advantages are indeed great. On the basis of System $\mathfrak{A}''$ it is possible to develop without any difficulty the most important parts of the arithmetic of real numbers, – such as the theory of the fundamental relations among numbers, the theory of the four arithmetical operations of addition, subtraction, multiplication, and division, the theory of linear equations, inequalities, and functions. The methods of inference to be applied here are of a very natural and quite elementary character; in particular, the axiom of continuity does not enter at all at this stage, it plays an essential role only when we go over to the "higher" arithmetical operations of raising to a power, of extracting roots and of taking logarithms, and it is indispensable for the proof of the existence of irrational numbers. No other system of axioms and primitive terms appears to be known that might furnish a more advantageous basis for an elementary and, at the same time, strictly deductive construction of the arithmetic of real numbers.

Exercises

1. Show that the set of primitive terms $\{\mathbb{R}, <, \cdot, 2\}$ form a model of System $\mathfrak{A}'$ and that, therefore, this system possesses at least two different interpretations within arithmetic.

2. Determine which of the following sets of numbers are densely ordered by the relation $<$:
   (a) The set of all natural numbers
   (b) The set of all integers
   (c) The set of all rational numbers
   (d) The set of all positive numbers
   (e) The set of all real numbers different from 0

3. In order to prove, on the basis of System $\mathfrak{A}'$, the existence of the number $\frac{1}{2}$, i.e. of a number z such that

$$z + z = 1,$$

we can proceed as follows. Let K be the set of all numbers x such that:

$$x + x < 1,$$

and, similarly, let L be the set of all numbers y such that:

$$1 < y + y.$$

We show first that the set K precedes the set L. Applying now the axiom of continuity, we obtain a number z separating the sets K and L. Next it can be shown that the number z can belong neither to K (otherwise a number $x \in K$ greater than z would exist) nor to L. From this we can conclude that z is the number looked for, in other words, that

$$z + z = 1$$

holds. Carry out in detail the proof sketched above.

4. Generalizing the procedure from the preceding exercise, prove the following theorem on the basis of System $\mathfrak{A}'$.

Theorem T. $\forall x \, \exists y \; (x = y + y)$

5. Replace, in System $\mathfrak{U}'$, Axiom $3'$ by Theorem T of the previous exercise. Show that the system of sentences obtained thereby is equipollent to System $\mathfrak{U}'$.

Hint: In order to derive Axiom $3'$ from the modified system, substitute "$x + z$" for "x" in Theorem T; in view of the hypothesis of Axiom $3'$, it can be shown easily that the number y fulfills the conclusion of the axiom.

6. Use the method of interpretation to show that, after omission of Axiom $1'$, System $\mathfrak{U}'$ becomes a system of mutually independent axioms.

7. Prove the following group-theoretical theorem:

Theorem. *If the class K is an Abelian group with respect to the operation O, then the operation O possesses exactly one unit element in the class K.*

8. Is the union of classes distributive with respect to intersection, and vice versa?

9. Determine which of the sets of numbers listed in Exercise 2 are fields with respect to addition and multiplication.

10. For those sets which were found to be fields from Exercise 9, determine which are ordered fields with respect to the given operation, and the relation $<$.

11. Show that the set $\mathbb{Z}_2$ is a field with respect to the operation $\oplus$ as defined in Exercise 5 of Chapter 9.

12. Show that the set $\mathbb{Z}_2$ is a field with respect to the operation $\cdot$.

13. How is it possible to define the symbol "1" with the help of multiplication?

14. The following theorem can be derived from the axioms of $\mathfrak{U}''$:

Theorem. $\forall x \, \exists y \; (0 < x \rightarrow x = y \cdot y)$.

Supposing this theorem to have been proved already, derive with its help from the systems of axioms of $\mathfrak{U}''$ the following theorem:

Theorem. $\forall x, y \, \exists z \; (x < y \leftrightarrow x + z \cdot z = y)$.

15. Does the theorem proven in the previous exercise justify a remark made in Section 11.5 concerning a possible reduction in the number of primitive terms of $\mathfrak{U}''$?

Notes

[1]The first axiomatic system for the entire arithmetic of real numbers was published by Hilbert in 1900; this system is relation to System $\mathfrak{U}''$. Before the year 1900, axiomatic systems for certain less comprehensive parts of arithmetic had been known; the first system of this kind relating to the arithmetic of natural numbers was given in 1889 by Peano (cf. endnote 1 of Chapter 6). Several axiomatic systems for arithmetic and various parts of it – and, in particular, the first axiomatic system for the arithmetic of complex numbers – were published by Huntington (cf. endnote 8 of Chapter 6).

[2]The axiom – in a slightly more complicated formulation – originates with the German mathematician R. Dedekind (1831-1916), whose researches have contributed greatly to the foundations of arithmetic and, especially, of the theory of irrational numbers.

Suggested Reading

In concluding this book, we should like to point out to the reader a number of works which may be of service in deepening and extending the knowledge acquired here. However, none of the works listed below offers a systematic and exhaustive treatment of all the problems upon which we have touched. As it is, the literature of the field in which we are interested is as yet comparatively poorly supplied with textbooks, and it is hard to name many books whose presentation combines comprehensibility with the required degree of exactitude.

G. Hunter. *Metalogic: An Introduction to the Metatheotry of Standard First Order Logic.* Berkeley: University of California Press, 1973.

This book is an excellent follow-up to the current text, and portions of this book have been added to discuss, in detail, the topics of completeness and consistency for propositional systems.

E.V. Huntington. “The Fundamental Propositions of Algebra”. In: *Monographs on Topics in Modern Mathematics Relevant to the Elementary Field.* Edited by J.W.A. Young. Monograph IV. Longmans, Green, and Company, 1911.

We recommend this work to the attention of those readers who are interested in the considerations contained in the last two chapters of the present book. They will find there a simple, precise, and clear presentation of methodological investigations into the axiomatic foundations of the arithmetic of real and complex numbers.

C.I. Lewis. *A Survey of Symbolic Logic.* Berkeley: University of California Press, 1918.

Though the systematic part of this book is somewhat outdated, its historical part can be warmly recommended, since it offers a great deal of interesting

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and instructive information on the development of modern logic.

C.I. Lewis and C.H. Langford. *Symbolic Logic.* 2nd Edition. Dover Publications, 1959.

This book contains much interesting and relevant material from various parts of symbolic logic and the methodology of deductive sciences, and, especially, from the domain of sentential calculus and its methodology. It cannot, however, serve as a systematic text of logic and methodology, since it fails to touch on many important topics belonging to these fields. The valuable historical material contained in Professor Lewis's work mentioned above has not been included in this book.

B. Russell. *Introduction to Mathematical Philosophy.* 2nd Edition. George Allen & Uniwn, Ltd., London, 1920.

This work gives a clear and easily intelligible presentation of the most important concepts of modern logic, especially those necessary for the establishment of mathematics as a part of logic. It treats among other things of many topics not discussed or only superficially touched upon in the present book, such as the theory of types or the problems connected with the axiom of infinity and the multiplicative axiom; it can serve as a preparatory text for the study of the work *Principia Mathematica* listed later.

J.W.A. Young. *Lectures on Fundamental Concepts of Algebra and Geometry.* The Macmillan Company, New York, 1911.

In this extremely informative little book the reader will find many interesting considerations and examples from the domain of the methodology of mathematics. Moreover, the reader may acquaint themselves here with some of the basic concepts of the general theory of sets.

J.H. Woodger. "The Technique of Theory of Construction". In: *International Encyclopedia of Unified Science.* Vol.2. No. 5. The University of Chicago Press, Chicago, 1939.

This short monograph will acquaint the reader by means of concrete example with the technique of constructing formalized deductive theories; it contains also a discussion of general methodological problems and interesting considerations as to the possibility and usefulness of applying deductive methods within empirical sciences. It can be warmly recommended, especially to readers interested in the last-mentioned problem.

The following two works are much more difficult:

R. Carnap. *The Logical Syntax of Language.* Kegan Paul, Trench, Trübner & Co. Ltd., London, 1937.

This book is interesting but not easy. As for its content, it corresponds to what we have called the methodology of deductive sciences, but in that wider conception which was discussed in Section 6.7. Emphasis is laid, however, not so much on a presentation of the results achieved in this field, as on the development of the conceptual apparatus. The systematic, deductive part of the book is treated rather sketchily, and it calls for quite a considerable amount of skill in abstract deductive thinking on the part of the reader, to enable the reader to fill the gaps which they will find and even, in places, to introduce some corrections n the arguments of the author. The book should be recommended, first of all, to those readers who are interested in the question of the significance of the methodological investigations for general philosophical problems; they will find numerous remarks on this subject throughout the text, and, moreover, more systematized considerations on it in the last part of the book.

A.N. Whitehead and B. Russell. *Principia Mathematica.* Vol.1–3. 2nd Edition. Cambridge University Press, 1927.

This work has already been quoted several times in the present book. It is undoubtedly the most representative work of modern logic, and a for the influence it has exerted it has been no less than epoch-making in the development of logical investigations. The purpose which the authors had in mind was to construct a complete system of logic which would provide a sufficient basis for the foundations of mathematics. The task was fulfilled in a very thorough and exhaustive manner (though not in ever detail complying with the very strictest requirements of present-day methodology). By most people the study of this work, which is written preponderantly in symbolic language, will not be found easy, but it is indispensable for anybody who desires to acquire a through knowledge of the conceptual apparatus of modern logic.

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