Math 2283 - Honors Logic

Homework - Chapter 6

Name: ______________________________

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1. *This is a continuation of Exercise 1 from the textbook.* Into the calculus of classes whose construction was outlined in the exercise 1 of the textbook, we may introduce the identity sign "=", defining it as follows:

Definition I. $K = L \overset{def}{\longleftrightarrow} K \subseteq L \wedge L \subseteq K$

From the axioms and theorems of Exercise I and the above definition derive the following theorems. Each theorem is worth 1 point.

Theorem XI. $K = K$

Hint: Let $L : K$ in Definition I and apply Axiom I.

Theorem XII. $K = L \to L = K$

Hint: In Definition I swap K and L, compare the sentence thus obtained with Definition I in its original formulation.

Theorem XIII. $(K = L \wedge L = M) \to K = M$

Hint: This theorem can be derived from Definition I and Axiom II.

Theorem XIV. $K \cup K = K$

Hint: In Definition I perform the substitution $K : K \cup K$ and $L : K$; apply Theorem I and Theorem III (with $L : K$).

Theorem XV. $K \cap K = K$

Hint: The proof is analogous to that of the preceding theorem.

Theorem XVI. $K \cup L = L \cup K$

Hint: By Theorem V we have:

$$K \cup L \subseteq L \cup K, \quad L \cup K \subseteq K \cup L$$

To these formulas apply Definition I.

Theorem XVII. $K \cap L = L \cap K$

Hint: The proof is similar to that of Theorem XVI.

Theorem XVIII. $K \cap (L \cup M) = (K \cap L) \cup (K \cap M)$

Hint: This theorem follows from: Definition I, Axiom V, and Theorem X.

Theorem XIX. $K \cup K' = U$

Hint: This theorem can be derived with the help of Definition I, from Axiom VI (with $K : K \cup K$), and Axiom VIII.

Theorem XX. $K \cap K' = \emptyset$

Hint: Apply Definition I, Axiom VII, and Axiom IX.

2. Let us assume that, into the system of the calculus of classes discussed in the previous exercise we introduce a new symbol "$\between$" denoting a certain relation between classes and defined as follows:

$$K \between L \overset{def}{\longleftrightarrow} \sim (K \subseteq L \vee L \subseteq K \vee K \cap L = \emptyset)$$

Is the relation defined in this way identical with any of the relations defined in Section 4.4?

3. The relation of disjointness between classes can be denoted by the symbol ")(". How can this symbol be defined within our system of the calculus of classes?

4. In addition to Theorems I and II, the following theorems can be derived from the axioms of Section 6.2:

Theorem III. $\forall\, x, y, z \in S\ \ ((x \cong y \wedge x \cong z) \rightarrow y \cong z)$

Theorem IV. $\forall\, x, y, z \in S\ \ ((x \cong y \wedge y \cong z) \rightarrow z \cong x)$

Theorem V. $\forall\, x, y, z, t \in S\ \ ((x \cong y \wedge y \cong z \wedge z \cong t) \rightarrow x \cong t)$

Give a strict proof that the following systems of sentences are equipollent, in the sense established in Section 6.4, to the system consisting of Axioms I and II (and that each might, therefore, be chosen as a new axiomatic system). We reproduce Axioms I and II, along with Theorems I and II for sake of completeness. Each of parts (a) and (b) are worth 1 point.

Axiom I. $\forall\, x \in S\ (x \cong x)$

Axiom II. $\forall\, x, y, z\ \in S\ \ ((x \cong z \wedge y \cong z) \rightarrow x \cong y)$

Theorem I. $\forall\, y, z \in S\ (y \cong z \rightarrow z \cong y)$

Theorem II. $\forall\, x, y, z \in S\ \ ((x \cong y \wedge y \cong z) \rightarrow x \cong z)$

(a) The system consisting of Axiom I and Theorem IV;
(b) The system consisting of Axiom I and Theorems I and V.

5. Along the lines of the remarks made in Section 6.2 formulate general laws of the theory of relations that represent a generalization of the results obtained in the preceding exercise.

Hint: These laws may, for instance, be given the form of equivalences, beginning with the words:

For a relation R to be reflexive and pseudotransitive in a class K, it is necessary and sufficient that ...

6. Applying the method of truth tables, we may introduce into sentential calculus new terms which were not discussed in Chapter 2. We can, for instance, introduce the symbol "$\downarrow$":

$$p \downarrow q \overset{def}{\longleftrightarrow} \text{ neither } p \text{ nor } q.$$

Construct the fundamental truth table for this function, which would comply with the intuitive meaning ascribed to the symbol "$\downarrow$".

7. Using the definition of "$\downarrow$" and the corresponding truth table from the previous exercise, verify, with the help of truth tables, that the following sentences are true and may be accepted as laws of sentential calculus.
(a) $\sim p \leftrightarrow (\,p \downarrow p\,)$
(b) $(\,p \vee q\,) \leftrightarrow [\,(\,p \downarrow q\,) \downarrow (\,p \downarrow q\,)\,]$
(c) $(\,p \rightarrow q\,) \leftrightarrow [\,[\,(p \downarrow p) \downarrow q\,] \downarrow [\,(p \downarrow p) \downarrow q\,]\,]$

8. *This is a continuation of Exercise 5 from the textbook.* If we are able to introduce defined terms into the system of sentential calculus described in Exercise 7 of the textbook, we have to assume a rule of definition. According to this rule (cf. Section 2.6), every definition has the form of an equivalence. The definiendum is an expression containing, besides sentential variables, only one constant, namely the term to be defined; no symbol may occur in this expression twice. The definiens is an arbitrary sentential function containing exactly the variables as the definiendum, and containing no constants except primitive terms and terms previously defined. Thus we may, for instance, accept the following definitions of the symbols "$\vee$" and "$\wedge$":

Definition I. $(p \vee q) \overset{def}{\longleftrightarrow} [\,(\sim p) \rightarrow q\,]$

Definition II. $(p \wedge q) \overset{def}{\longleftrightarrow} [\,\sim [\,(\sim p) \vee (\sim q)\,]\,]$

From the above definitions and the axioms and theorems of Exercise 10 deduce the following theorems with the help of the Rule of Detachment. Each theorem is worth 1 point.

Theorem XI. $[\,(\sim p) \rightarrow q\,] \rightarrow (p \vee q)$

Hint: In Axiom V use the instance $p : (p \vee q)$ and $q : [\,(\sim p) \rightarrow q\,]$; compare the sentence thus obtained with Definition I and apply the Rule of Detachment.

Theorem XII. $p \vee (\sim p)$

Hint: This theorem can be derived from instances of Theorems XI and I and an application of the Rule of Detachment.

Theorem XIII. $p \to (p \vee q)$

Hint: The proof is based upon Axiom III and Theorem VI and XI, and is quite similar to that of Theorem II (use two distinct instances of Axiom III only).

Theorem XIV. $(p \wedge q) \to [\sim [(\sim p) \vee (\sim q)]]$

Hint: The proof, which is based upon Axiom IV and Definition II, is analogous to that of Theorem XI.

Theorem XV. $[\sim [\sim (p \wedge q)]] \to [\sim [(\sim p) \vee (\sim q)]]$

Hint: This proof is based upon Axiom III and Theorems VIII and XIV, and is similar to that of Theorem II. In Theorem VIII use the instance $p : (p \wedge q)$ and compare the consequent of the implication thus obtained with the antecedent of Theorem XIV.

Theorem XVI. $[(\sim p) \vee (\sim q)] \to [\sim (p \wedge q)]$

Hint: Find an instance of Axiom VII such that the antecedent of the resulting implication will be Theorem XV.

Theorem XVII. $(\sim p) \to [\sim (p \wedge q)]$

Hint: The proof is again analogous to that of Theorem II. In Theorem XIII use the instance $p : (\sim p)$ and $q : (\sim q)$; compare the resulting sentence with Theorem XVI.

Theorem XVIII. $(p \wedge q) \to p$

Hint: Derive this theorem from Axiom VII and Theorem XVII.

9. One of the laws of sentential calculus is the following:

$$(p \wedge \sim p) \to q$$

On the basis of this logical law, establish the following methodological law:

If the axiomatic system of any deductive theory which presupposes sentential calculus is inconsistent, then every sentence formulated in the terms of this theory can be derived from that system.

10. It is known that the following methodological law holds:

If the axiomatic system of a deductive theory is complete, and if any sentence which can be formulated but not proved within that theory is added to the system, then the axiom system extended in this manner is no longer consistent.

Why is this the case?