$$\int_{C} f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b \big(u(z(t)) + i\,v(z(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt$$

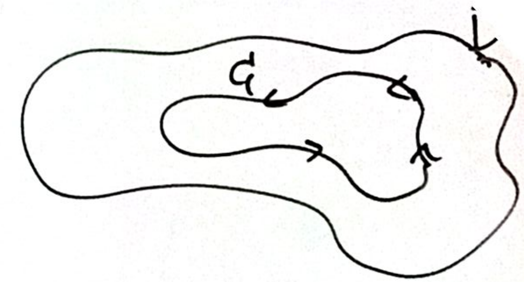
$$= \int_a^b u x' - v y'\,dt + i \int_a^b v x' + u y'\,dt$$

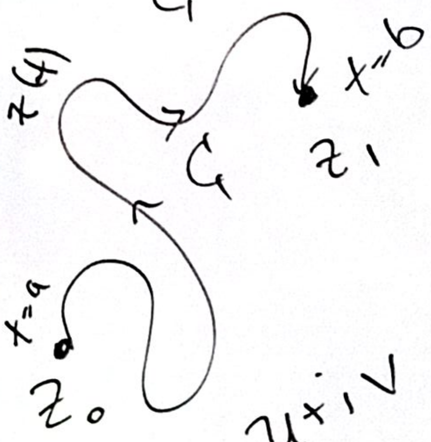
$$= \int_C u\,dx - v\,dy + i \int_C v\,dx + u\,dy$$

$$\parallel$$

$$\iint_R -v_x - u_y\,dx\,dy + i \iint_R u_x - v_y\,dx\,dy$$

$$\int_{C_1} f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$





$$= \int_a^b \big(u(z(t)) + i\,v(z(t))\big)\big(x'(t) + i\,y'(t)\big)\,dt$$

$f(z) = u + iv$

$$= \int_a^b u x' - v y'\,dt + i\int_a^b v x' + u y'\,dt$$

$dx = x'\,dt$

$\frac{dx}{\cancel{dt}}\,\cancel{dt} = x'$

Green's Theorem

$$= \int_{C_1} u\,dx - v\,dy + i\int_{C_1} v\,dx + u\,dy \quad = 0$$

$$\parallel$$

$$\iint_R \underbrace{-v_x - u_y}_{0}\,dx\,dy + i\iint_R \underbrace{u_x - v_y}_{0}\,dx\,dy$$

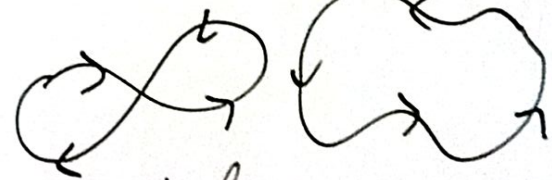
Theorem: Let f be analytic in a simply connected domain D.

$$\int_{C_1} \frac{1}{z-z_0}\,dz$$

If C_1 is a simple closed contour in D, then

$$\int_{C_1} f(z)\,dz = 0$$

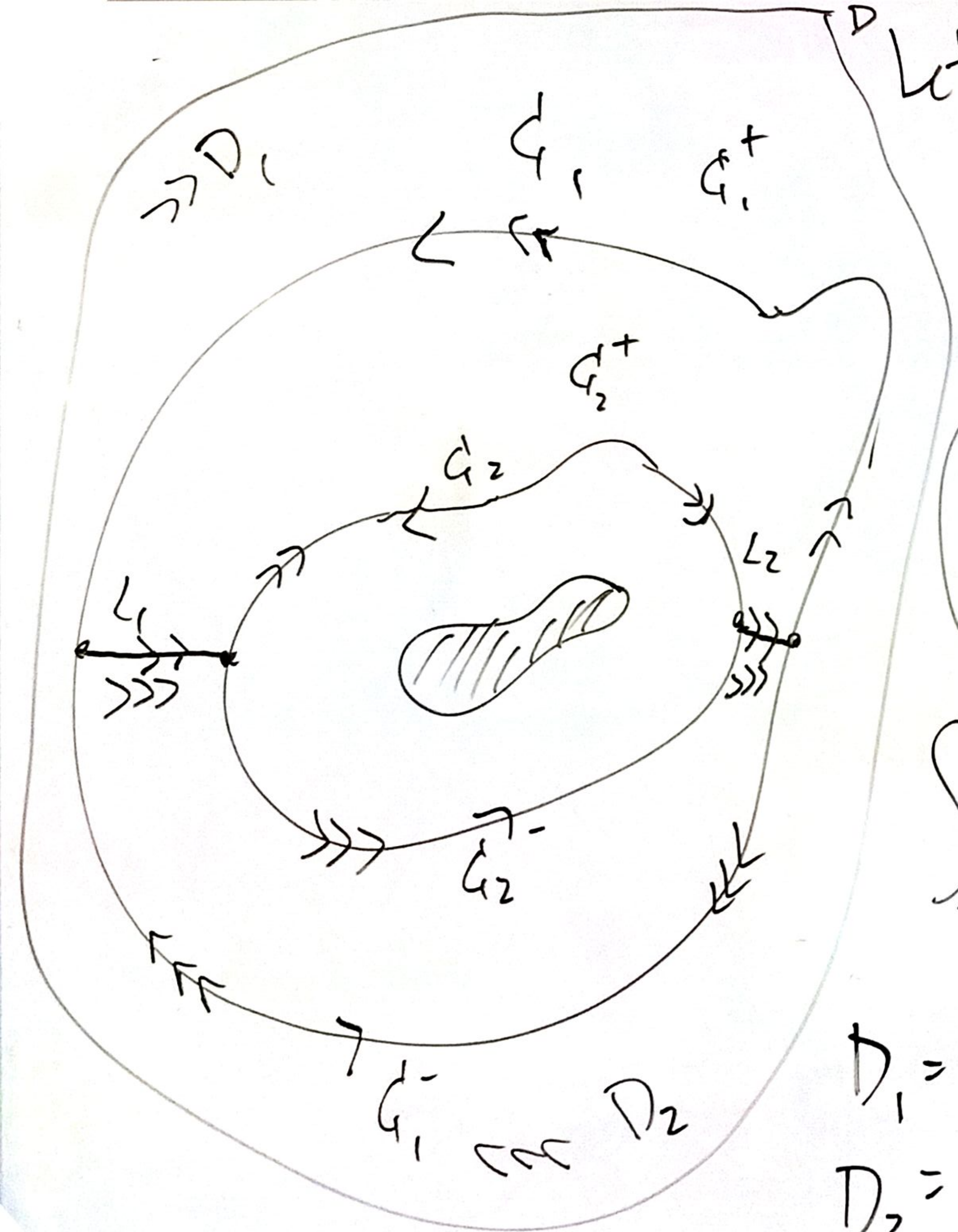
$z(t_0) = z(t_1) = z_0$



D simply connected – all one piece, no holes

connected – for $\forall z_0, z_1 \in D$, a path $\gamma(t)$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z_1$, w/ $\gamma(t) \subset D$ exists

and there are no holes



Let $f(z)$ be analytic on D

Claim:

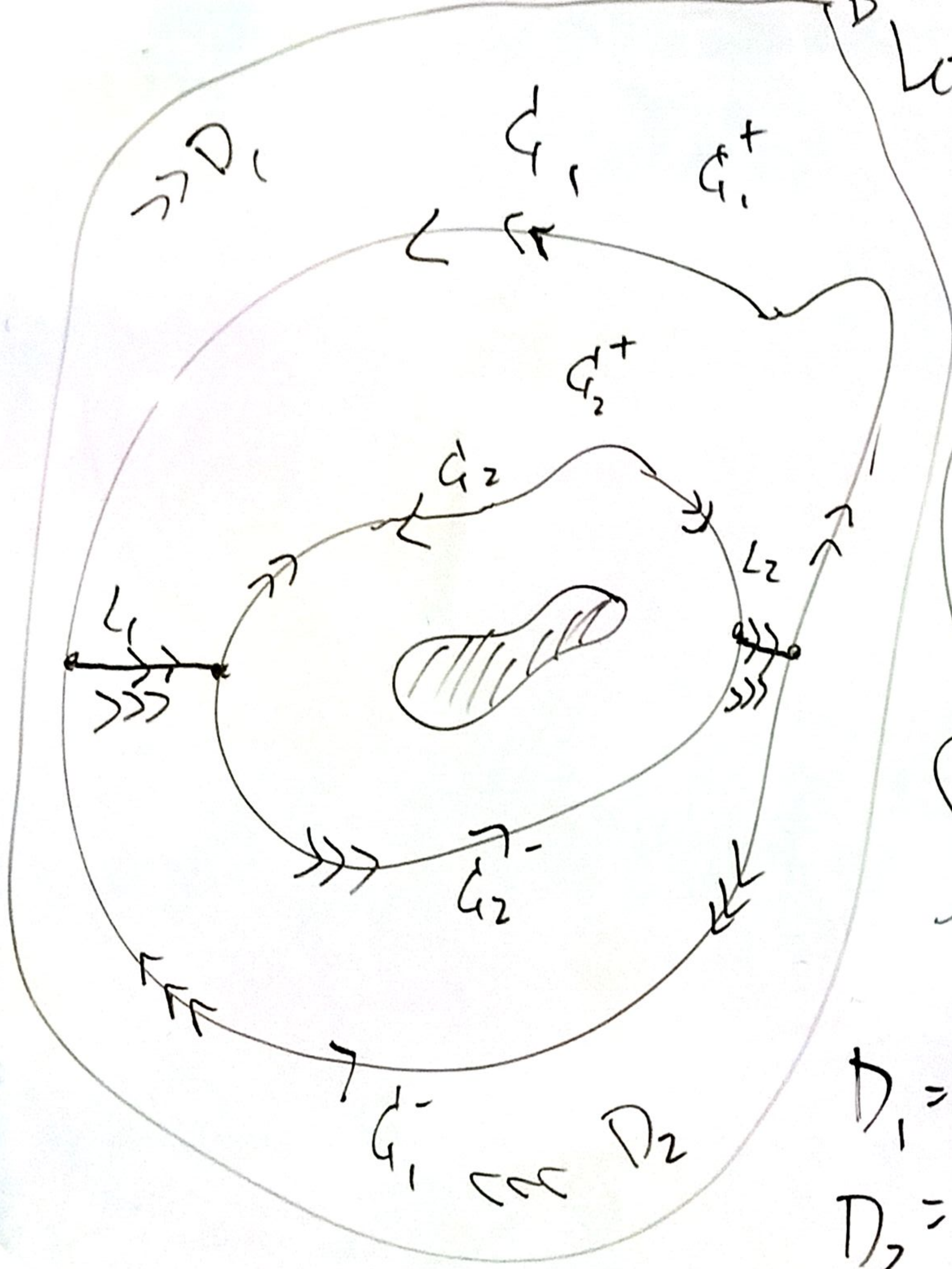
$$-C_2^+ + C_1^+ = C_2^- - C_1^-$$

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

$$\int_{D_1} f(z)\,dz = 0 = \int_{D_2} f(z)\,dz$$

$$D_1 = \cancel{L_1} - C_2^+ + \cancel{L_2} + C_1^+$$

$$D_2 = \cancel{L_1} + C_2^- + \cancel{L_2} - C_1^-$$



Let $f(z)$ be analytic on D

$$-C_2^+ + C_1^+ = C_2^- - C_1^-$$

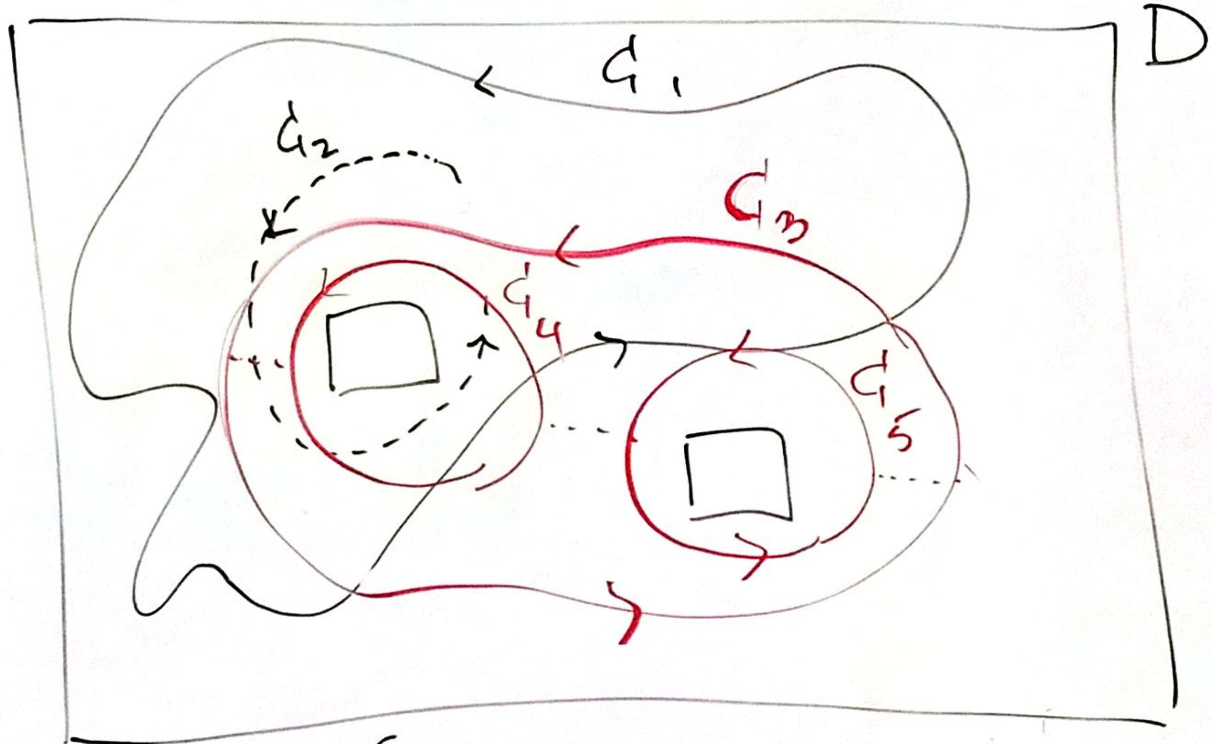
Claim:

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

$$\int_{D_1} f(z)\,dz = 0 = \int_{D_2} f(z)\,dz$$

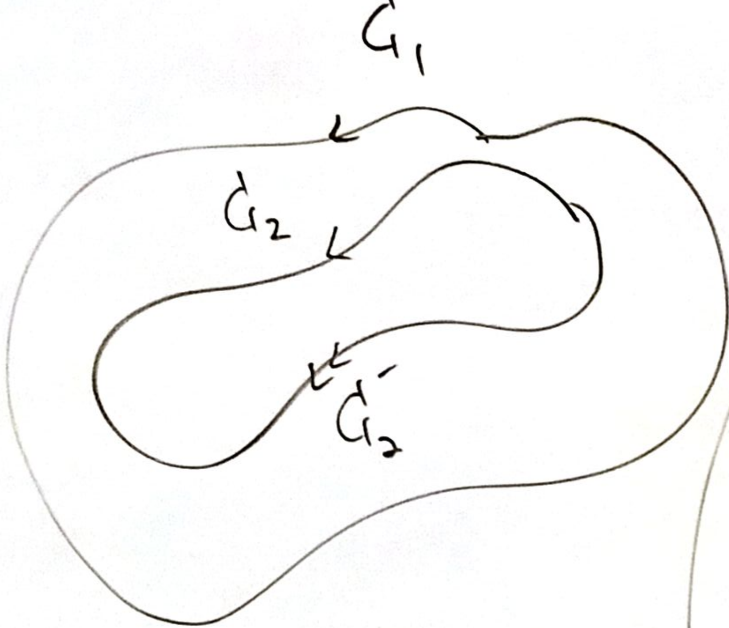
$$D_1 = \cancel{L_1} - C_2^+ + \cancel{L_2} + C_1^+$$

$$D_2 = \cancel{L_1} + C_{12}^- + \cancel{L_2} - C_1^-$$



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\int_{C_3} f(z)\,dz = \int_{C_4} f(z)\,dz + \int_{C_5} f(z)\,dz$$



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\int_{C_2^-} f(z)\,dz = -\int_{C_2} f(z)\,dz$$

$$= -\int_{C_2^-} f(z)\,dz$$

Cauchy-Goursat

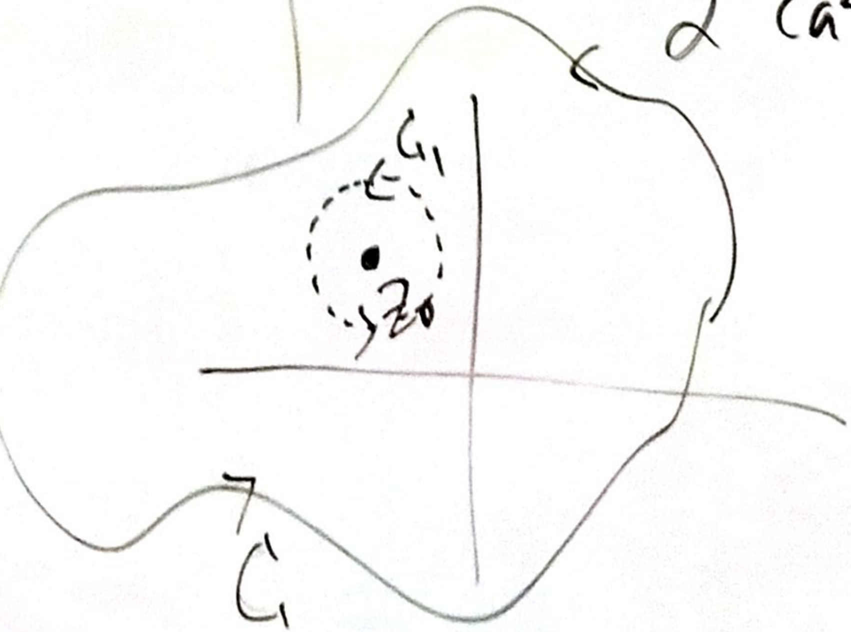
$$f(z) = \frac{1}{z - z_0}$$

$$\int_{C_1} f(z)\,dz$$

C_1 is simply closed counter clockwise contour

2 cases: (1) $z_0 \in \text{int}(C_1)$

(2) $z_0 \notin \text{int}(C_1)$



$$\int_{C_1} f(z)\,dz = 0$$ by C.G.

Cauchy - Goursat

$$\int_C \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \in \operatorname{int}(C_1) \\ 0 & \text{if } z_0 \notin \operatorname{int}(C_1) \end{cases}$$

$$\int_{C_1} (z-z_0)^n\,dz = 0 \quad \text{if } n \neq -1 \quad n \in \mathbb{Z}$$

regardless of $z_0 \in \operatorname{int}(C_1)$ or $z_0 \notin \operatorname{int}(C_1)$

$$\int_{C_1} f(z)\,dz = 2\pi i \cdot a_{-1}$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$