6.3 || $\displaystyle\int_{C_1^+(0)} |z|^2 e^z \, dz$

$z(\theta): e^{i\theta}, \quad \theta: -\pi \to \pi$

$|z(\theta)| = 1 \quad \forall \theta \in (-\pi, \pi)$

$dz = i e^{i\theta} d\theta$

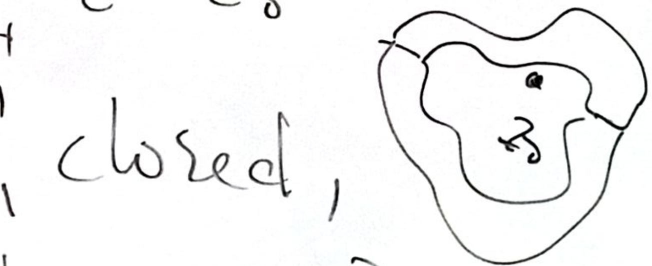
$$\int_{-\pi}^{\pi} 1 \cdot e^{e^{i\theta}} \, i e^{i\theta} \, d\theta$$

6.5 $$\frac{1}{2\pi i}\int_{C_R^+(z_0)} \frac{1}{z-z_0}\,dz = 1$$

let $f(z)$ be analytic on a simply connected domain D, $C_R^+(z_0) \subseteq D$

let $z_0 \in D$, $$\frac{1}{2\pi i}\int \frac{f(z_0)}{z-z_0}\,dz = f(z_0)$$

Claim: $$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_0}\, dz$$



where C_1 is any simple, closed, positively oriented contour in D w/ z_0 on the interior of C_1.

(1) Contour can be deformed to a circle centered at $z = z_0$.

We can then consider:

$$\frac{1}{2\pi i}\int_{C_R^+(z_0)} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_R^+(z_0)} \frac{f(z_0)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_R^+(z_0)} \frac{f(z)-f(z_0)}{z-z_0}\,dz \overset{?}{=} 0$$

we can use ML inequality

$$|z-z_0| = R$$

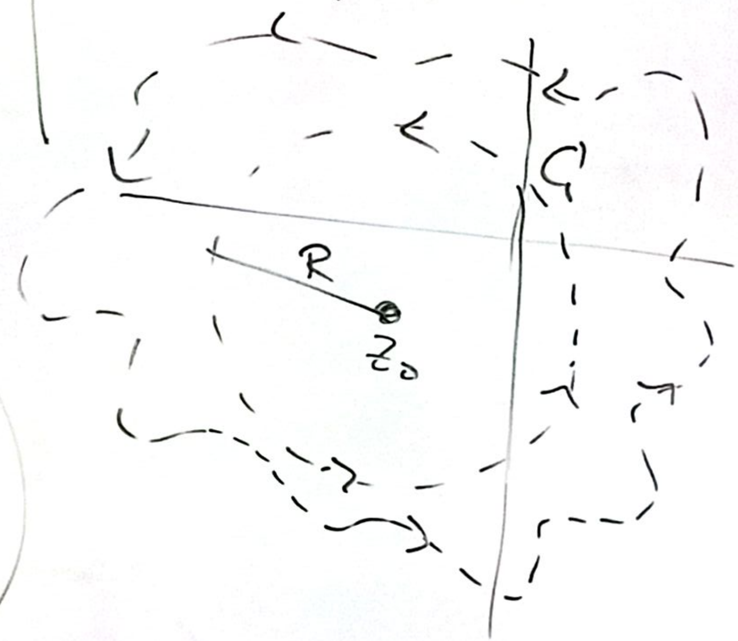
$$L = 2\pi R$$

If $f(z)$ is analytic given $\delta>0$

$$|f(z)-f(z_0)| < \varepsilon \quad \text{if} \quad |z-z_0|<\delta$$

$$\left|\frac{1}{2\pi i}\int_{C_\delta^+(0)} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right|$$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



$$\le \frac{1}{2\pi}\cdot\underbrace{2\pi\delta}_{L}\cdot\underbrace{\frac{\varepsilon}{\delta}}_{M}$$

$$= \varepsilon$$

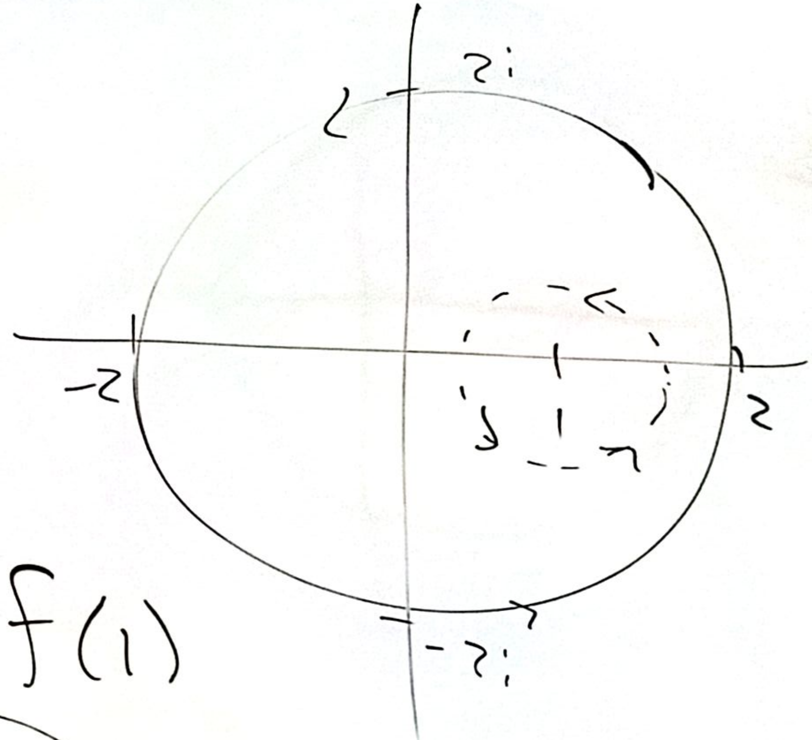
ε arbitrary Q.E.D.

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R^+(z_0)} \frac{f(z)}{z-z_0}\,dz$$

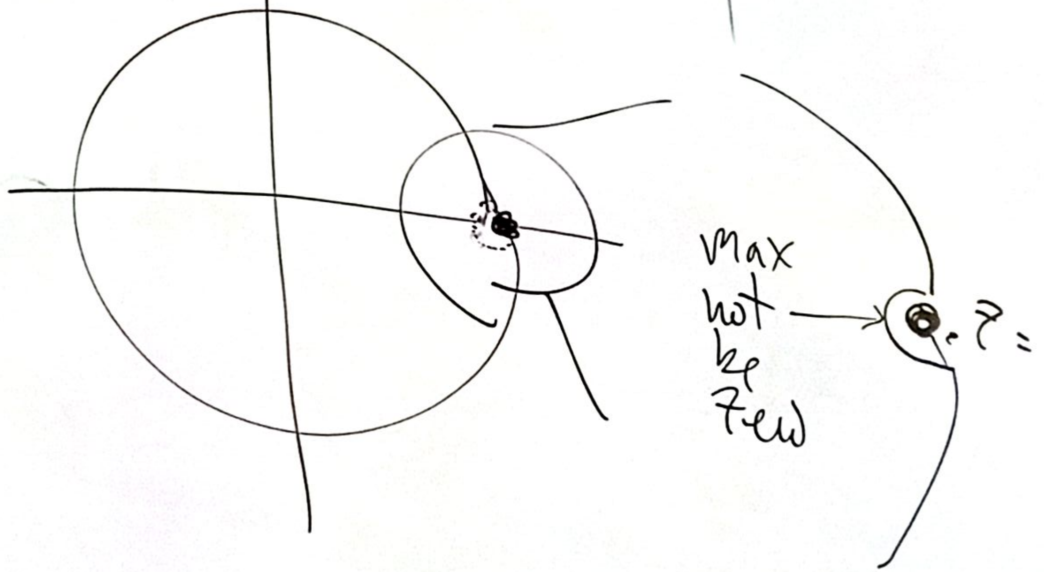
Ex 6.21

$$\int_{C_2^+(0)} \frac{e^z}{z-1}\,dz$$

$z_0 = 1$



$$= 2\pi i \cdot e^1 = 2\pi i \cdot f(1)$$



Ex 6.23

$$\int_{C_1(0)} \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

where C_1 contains z_0 only point not analytic at.

$$\int_{C_1^+(0)} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z-2)}\,dz$$

$$= \frac{1}{2}\int_{C_1^+(0)} \frac{\underbrace{e^{i\pi z}/(z-2)}_{f(z)}}{z-1/2}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i \cdot f(1/2) = \pi i \cdot \frac{e^{i\pi/2}}{-3/2}$$

$$= \frac{2}{3}\pi$$

Ex 6.23

extended Cauchy–Goursat
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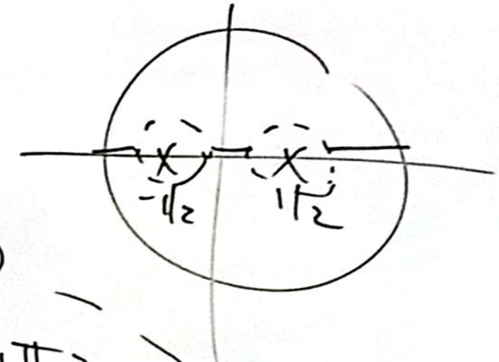

$$\int_{C_1^+(0)} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z+\frac{1}{2})}\,dz$$

$f(z)$ $\qquad\qquad\qquad\qquad$ $f(z)$

$$= \int_{C_{1/2}^+(-1/2)} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z+\frac{1}{2})}\,dz + \int_{C_{1/2}^+(1/2)} \frac{e^{i\pi z}}{2(z-\frac{1}{2})(z+\frac{1}{2})}\,dz$$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

$f(z)$ analytic on D, w/ $C \subseteq D$ simple closed + - oriented contour. D simply connected.

Thm 6.12

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

Cauchy's Integral Formula for derivatives

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi$$

$$f(z_0) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z - z_0}\,dz$$

$f(z)$ analytic on D, w/ $C \subseteq D$ simple closed +-oriented contour. D simply connected.

Thm 6.12

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C} \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi}$$

Cauchy's Integral Formula for derivatives

$$f(z) = \frac{1}{2\pi i}\int_{C} \frac{f(\xi)}{\xi - z}\,d\xi$$