6.6 From calculus:

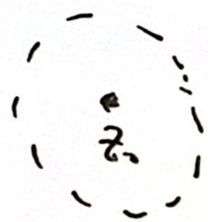
$$F(x) = \int_{x_0}^{x} f(t)\,dt$$ is a well defined function, if $f(t)$ is integrable on the interval $[x_0, x]$

Furthermore, $F'(x) = f(x)$

The same holds for $F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$

6.6 Morera's Theorem:

If $f(z)$ is continuous on simply connected domain D and $\int f(z)\,dz = 0$ for every closed contour C, C in D, then $f(z)$ is analytic in D.



6.6 Gauss's Mean Value Theorem

If $f(z)$ is analytic on a S.C.D. D

w/ $C_R(z_0) \subseteq D$

then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta$

i.e. we can "average $f(z)$ in a circle about $z = z_0$ to get $f(z_0)$

6.6 Let $z = z_0 + Re^{i\theta}$ $\theta: 0 \to 2\pi$

$dz = iRe^{i\theta}\,d\theta$

$$\int_{C_R(z_0)} f(z)\,dz = \int_0^{2\pi} f(z_0 + Re^{i\theta})\, iRe^{i\theta}\, d\theta$$

Cauchy:

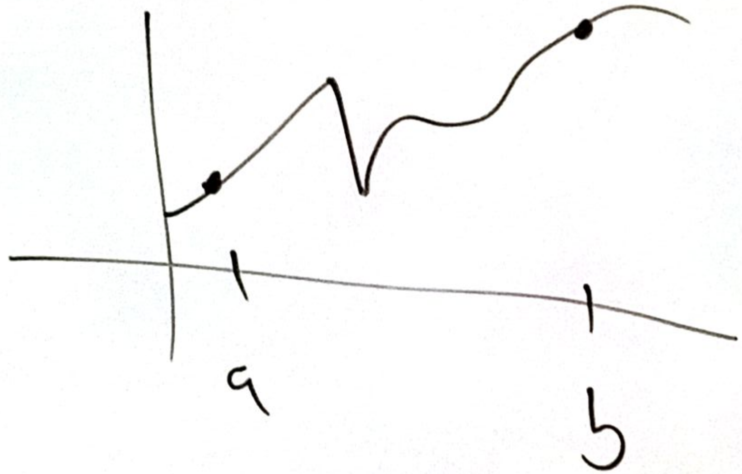
$$f(z_0) = \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(z)}{z - z_0}\,dz$$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{\cancel{Re^{i\theta}}}\,\cancel{iRe^{i\theta}}\,d\theta$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$$

6.6 From calculus

a continuous function on a closed, bounded interval attains a maximum and minimum value.



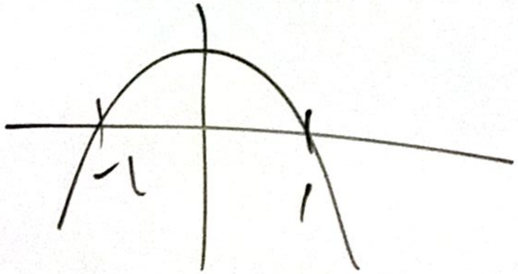
6.6 Max modulus Principle

Let $f(z)$ be analytic on S.C.D. D then $|f(z)|$ cannot attain a maximum on the interior of D.

i.e. there is no point $z_0 \in D$ s.t.

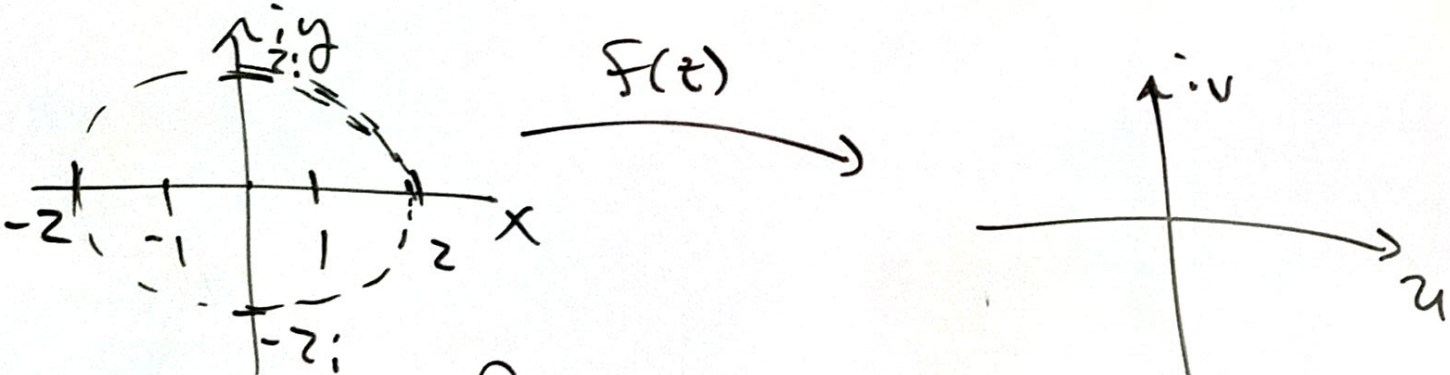
$$|f(z)| \leq |f(z_0)| \quad \forall z \in D$$

6.6 $f(x) = 1 - x^2$



on $[-1, 1]$ has max at $x = 0$.

What about $f(z) = 1 - z^2$



$f(x+iy) = 1 - (x+iy)^2 = 1 - (x^2 - y^2 + i2xy)$

$|f(z)|^2 = (1 - x^2 + y^2)^2 + (4x^2y^2)$

6.6 Max modulus Principle

Let $f(z)$ be analytic and nonconstant on a bounded domain D. If $f(z)$ is continuous on R, which is the union of D and its boundary, then $|f(z)|$ assumes its max value at point(s) on the boundary of D.

6.6 Cauchy's Inequalities

Let f be analytic on S.C.D. D

s.t. $C_R(z_0) \subseteq D$

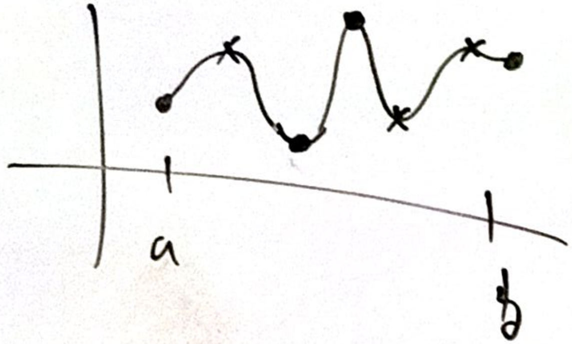
If $|f(z)| \leq M \quad \forall z \in C_R(z_0)$

then $\left| f^{(n)}(z_0) \right| \leq \dfrac{n!\, M}{R^n}$

---

start: $f^{(n)}(z_0) = \dfrac{n!}{2\pi i} \displaystyle\int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

6.6 Global max/mins



global max/mins are relative max/mins but not always vice versa

$f(x) = x$ has no global max/min on $\mathbb{R}$

$$\left.\begin{aligned} f(x) &= \sin(x) \\ &= \tanh^2(x) \\ &= e^{-x^2} \end{aligned}\right\} \text{ have global max or mins}$$

6.6 The only bounded entire function on $\mathbb{C}$ is a constant function.