$$\{x_n\}_{n=0}^{\infty} \longrightarrow x$$

$$\lim_{n\to\infty} x_n = x \longleftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. }$$

$$|x_n - x| < \varepsilon \quad \forall n \geq N$$

Cauchy convergence

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N$$

$$|x_n - x_m| < \varepsilon$$

let $\{f_n(x)\}$ be a sequence of functions. What does it mean for $\{f_n(x_0)\} \to \underbrace{f(x_0)}_{x}$?

it's the same thing

$\{x_n\}$

But if $f_n(x)$ has a domain D, the rate of convergence might be different for each $x_0 \in D$

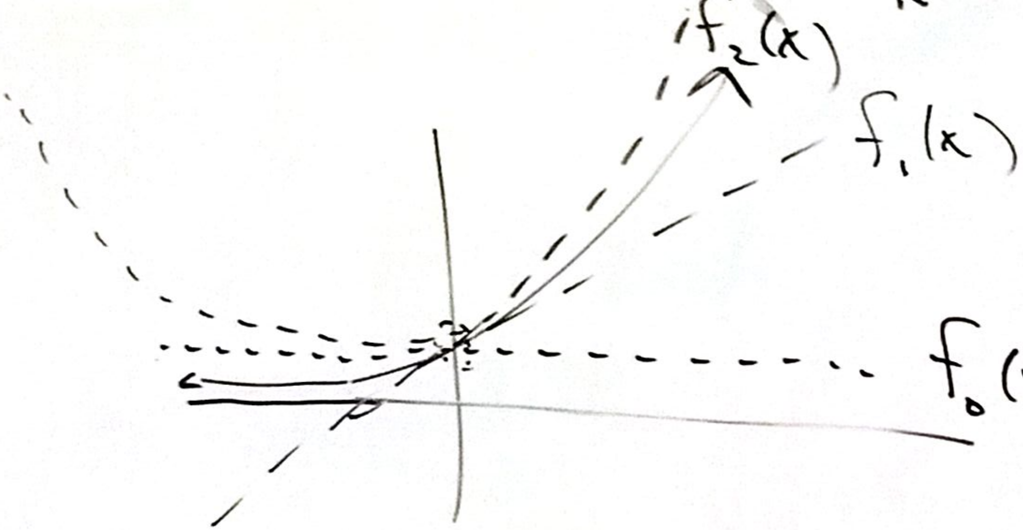
Ex: $f(x) = e^x$

define $f_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$

$f_0(x) = 1$

$f_1(x) = 1 + x$

$f_2(x) = 1 + x + \frac{x^2}{2}$



the further away from $x=0$ we go, the more error there is for a fixed n.

$$f(x) = e^x = \lim_{n\to\infty} f_n(x).$$

Convergence $\{f_n(z)\} \to f(z)$ if $\forall z_0 \in$ domain of $f(z)$

$$\lim_{n\to\infty} f_n(z_0) = f(z_0)$$

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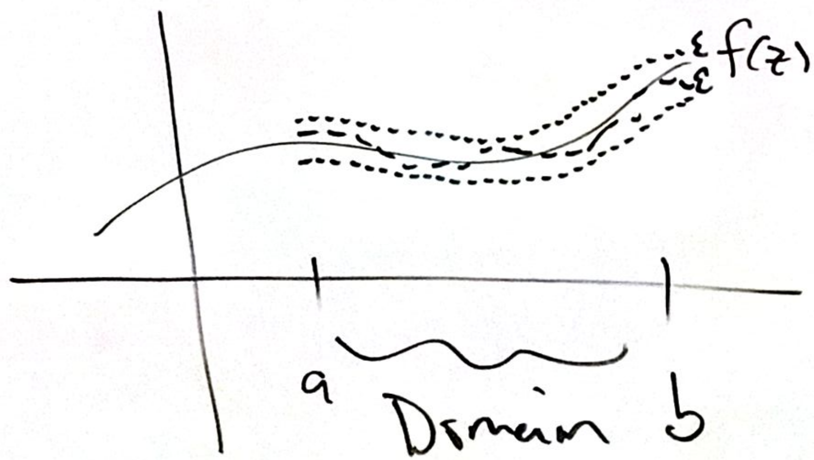
uniform convergence

We say $\{f_n(z)\}$ converges uniformly to $f(z)$, if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|f_n(z) - f(z)| < \varepsilon \quad \forall n \geq N,\ z \in \text{Domain of } f(z)$$



For us, given $f(z)$, complex valued function, we are going to study sequences of the form $S_n(z) = \sum_{k=0}^{n} c_k (z-\alpha)^k$

which are the n^{th} Taylor series approximations to $f(z)$, $c_k \in \mathbb{C}$, $\alpha \in \mathbb{C}$.

We want uniform convergence of $S_n(z)$ to $f(z)$

Thm 7.1 Weierstrass M-test

Suppose $\sum_{k=0}^{\infty} u_k(z)$ has the property

that $|u_k(z)| \le M_k \quad \forall z \in T$, the

domain of $u_k(z) \ \forall k$.

If $\sum_{k=0}^{\infty} M_k$ converges, then

$\sum_{k=0}^{\infty} u_k(z)$ converges uniformly $\forall z \in T$

Show $\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ converges uniformly on $\overline{D_1(0)}$

$$u_k(z) = \frac{1}{k^2} z^k$$

need to find M_k s.t. $|u_k(z)| \le M_k \;\forall z \in \overline{D_1(0)}$

and $\sum_{k=1}^{\infty} M_k < \infty$. How do we pick M_k?

$$|u_k(z)| = \left|\frac{1}{k^2} z^k\right| = \frac{1}{k^2}|z|^k \le \frac{1}{k^2} = M_k$$

Q: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge? yes, to $\frac{\pi^2}{6}$

Domain very much matters!

$\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ has a radius of convergence $\rho > 0$. Then for each r, $0 < r < \rho$, the series converges uniformly on $\overline{D_r(\alpha)}$

<u>Thm 7.3</u> $\{S_k\}$ a sequence of continuous functions on T, w/ $C \subset T$. If $\{S_k\}$ converges uniformly on T to $f(z)$,

(i) $f(z)$ is continuous on T

(ii) $\lim_{k \to \infty} \int_C S_k(z)\,dz$

$= \int_C \lim_{k \to \infty} S_k(z)\,dz = \int_C f(z)\,dz$

$$\int_{C_R^+(z_0)} (z-z_0)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

if $f(z)$ has a power series expansion

$$f(z) = \lim_{n\to\infty} S_n(z), \quad S_n(z) = \sum_{k=0}^{n} c_k (z-a)^k$$

whose radius of convergence is $\rho > 0$.

let $R < \rho$

$$\int_{C_R^+(0)} f(z)\,dz = \int_{C_R^+(0)} \lim_{n\to\infty} S_n(z)\,dz = \lim_{n\to\infty} \int_{C_R^+(0)} \sum_{k=0}^{n} c_k (z-a)^k \, dz$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \int_{C_R^+(0)} c_k (z-a)^k \, dz = 0$$

$$f(z) = \frac{e^{z}}{z^{3}} = \frac{1}{z^{3}} \sum_{k=0}^{\infty} \frac{1}{k!} z^{k}$$

$$= \frac{1}{z^{3}} \left(1 + z + \frac{z^{2}}{2} + \frac{z^{3}}{3!} + \cdots \right)$$

$$= \frac{1}{z^{3}} + \frac{1}{z^{2}} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^{2}}{5!} + \cdots$$

$$\int_{C_1^{+}(0)} f(z)\,dz$$