Series representation

and uniform convergence

$$\{S_n(z)\} \to f(z) \quad \forall z \in T$$

For series, we get uniform convergence on $D_R(\alpha)$, for $0<R<\rho$ where ρ is the radius of convergence for $\lim_{n\to\infty} S_n(z)$

Taylor series for $f(z)$ at $z=\alpha$:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

if $\alpha=0$, we call this a

Maclaurin series

Taylor's Theorem: Suppose $f(z)$ is analytic on a domain G, w/ $D_R(\alpha) \subseteq G$. Then the Taylor series for $f(z)$ converges to $f(z)$ $\forall z \in D_R(\alpha)$

for $0 < r < R$, the convergence is uniform on $\overline{D_r(\alpha)}$.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

converges for $|z| < 1$

what happens if $|z| > 1$?

$$\frac{1}{z} \frac{1}{\frac{1}{z} - 1} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}}$$

$$= -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= -\frac{1}{z} \sum_{k=-\infty}^{0} z^{-k}$$

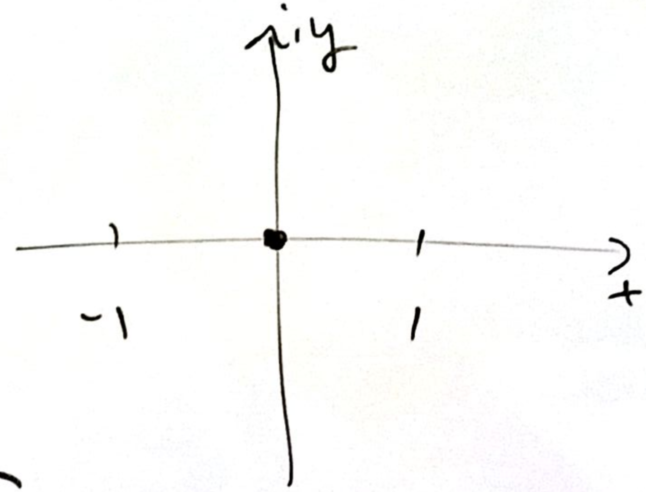
converges for $|z| > 1$

Given a Taylor Series on domain G,
then the series converges uniformly for
all values of R s.t. $f(z)$ is analytic
on $D_R(\alpha)$ where $0 < R < r$, here
r is the radius of convergence for the series.
If $f(z)$ is singular at $z = z_0$
then $R < |\alpha - z_0|$

$$f(z) = \frac{1}{z^2 - 1} = -\frac{1}{1-z^2}$$

let's say there is a Maclaurin series for $f(z)$. What is it's maximum radius of convergence?

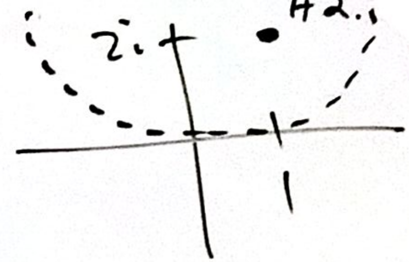
Max radius is $r=1$



since singularities are a distance 1 from $a=0$

$$f(z) = \sum_{k=0}^{\infty} -1 \cdot z^{2k} \quad \text{for } |z|<1$$

$f(z) = \frac{1}{1-z}$, $\alpha = 1+2i$



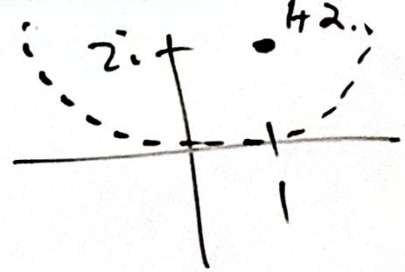
find Taylor series & radius of convergence

$$\frac{1}{1-z} = \frac{1}{1-1-2i-(z-(1+2i))}$$

$$= \frac{1}{-2i-(z-(1+2i))} \quad +1+2i$$

$$= \frac{1}{-2i}\,\frac{1}{1-\frac{z-(1+2i)}{-2i}}$$ new z

$$f(z) = -\frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^{k}(z-(1+2i))^{k}$$



$$f(z) = \frac{1}{1-z}, \quad \alpha = 1+2i$$

find Taylor series & radius of convergence

$$\frac{1}{1-z} = \frac{1}{1-1-2i-(z-(1+2i))}$$

$$\left|\frac{z-(1+2i)}{2i}\right| < 1$$

$$= \frac{1 \quad +1+2i}{-2i-(z-(1+2i))}$$

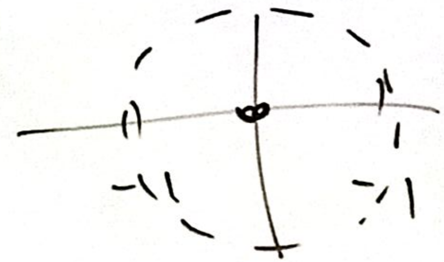
$$|z-(1+2i)| < 2$$

$$= \frac{1}{-2i} \cdot \frac{1}{1-\frac{z-(1+2i)}{-2i}}$$

new z

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^{k+1} (z-(1+2i))^k$$

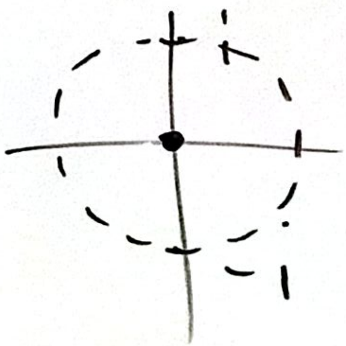
$$f(z) = \frac{1}{z^2-1} = \frac{-1}{1-z^2}$$



$$= \sum_{k=0}^{\infty} -1 \cdot z^{2k}$$

converges for $|z|<1$ uniformly

$$g(z) = \frac{1}{z^2+1} = \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$



but we still only get convergence for $|z|<1$

since there are singularities at $z = \pm i$