$$\sum_{k=0}^{\infty} \frac{z^k}{z^{2k}+1} \qquad \overline{D_r(0)}, \quad 0<r<1$$

$$z = \rho e^{i\theta}$$

$$\Big|\quad\Big| \le \frac{1}{1-\rho^2} \sum_{k=0}^{\infty} |z|^k$$

$$z^2 = \rho^2 e^{i2\theta}$$

$$= \frac{|z|}{1-|z|} \cdot \frac{1}{1-\rho^2}$$

$$z^{2k} = \rho^{2k} e^{i2k\theta}$$

$$\underline{M_k} = \frac{|z|^k}{1-\rho^2}$$

$$\text{if } \theta = \pi/2, \quad z^{2k} = \rho^{2k} e^{i2k\cdot\frac{\pi}{2}}$$

$$\text{if } k=1, \quad z^2 = -\rho^2 \longrightarrow z^2+1 = 1-\rho^2$$

$$\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k} \qquad D = \{ z \mid z \geq 2 \}$$

$$u_k(z) = \frac{1}{(z^2-1)^k}$$

Identity 1-24 $\quad |z^2-1| \geq |z^2| - |-1|$

$$\geq |z|^2 - 1 \geq 2^2 - 1$$

$$\geq 3 \quad \forall z \in D$$

$$M_k = \frac{1}{3^k}$$

Thus we get uniform convergence!

$$\sum_{k=0}^{\infty} \frac{1}{k^2} z^k \qquad \overline{D_1(0)}$$

$$|u_k| \leq \frac{1}{k^2} \rightarrow \text{Uniform convergence!}$$

4. 7.3 $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$

$\exists n \geq N$ and $z_0 \in T$ s.t.

$$|S_n(z_0) - f(z_0)| \geq \varepsilon$$

Let $\varepsilon > 0$, define, for $n \in \mathbb{N}$, $z_n = \varepsilon^{1/n} \in D_1(0)$

$$|S_n(z_n) - f(z_n)| = \left| \frac{1-\varepsilon}{1-\varepsilon^{1/n}} - \frac{1}{1-\varepsilon^{1/n}} \right|$$

$$= \frac{\varepsilon}{1-\varepsilon^{1/n}} > \varepsilon$$

5. Thm 7.2 – $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$

power series

why can we **not** argue using 7.2 to prove the geometric series converges uniformly on $D_1(0)$?

$$\sum_{k=0}^{\infty} |c_k (z-\alpha)^k| \leq \sum_{k=0}^{\infty} |c_k| r^k$$

we need ⟶ to converge

$c_k \equiv 1$ $\quad \sum M_k$ to converge

6. $\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$

$$\int_0^z \cos(\xi)\, d\xi = \sin(z) - \sin(0) = \sin(z)$$

by Cor. 7.2

$$\int_0^{z_0} \cos(\xi) d\xi = \sum_{k=0}^{\infty} \int_0^{z_0} (-1)^k \frac{z^{2k}}{(2k)!} dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^{z_0} z^{2k} dz$$

$$\sin(z_0) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{(2k)!} \frac{z^{2k+1}}{2k+1}}_{(2k+1)!} \Bigg|_0^{z_0} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

1. (a) $\sinh(z)$

$\frac{d}{dz}\sinh(z) = \cosh(z)$

$\frac{f^{(n)}(0)}{n!}(z-0)^n$

$\frac{d}{dz}\cosh(z) = \sinh(z)$

$\cosh(0) = 1$

$\sinh(0) = 0$

$0 + z + 0 \cdot \frac{z^2}{2!} + 1 \cdot \frac{z^3}{3!} + 0 \cdot \frac{z^4}{4!} + 1 \cdot \frac{z^5}{5!} + \ldots$

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{1}{k!} z^k - \sum_{k=0}^{\infty} \frac{1}{k!} (-z)^k\right]$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!}\left(z^k - (-z)^k\right)$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!}\left(z^k - (-1)^k z^k\right)$$

if k even, cancel
if k odd, combine:

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \cdot 2\, z^{2k+1}$$