7.3 Laurent Series Representations

Taylor Series:

$$f(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

$$c_k = \frac{f^{(k)}(\alpha)}{k!} = \frac{1}{k!} \int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{k+1}} \, dz$$

Q: what happens at singularities?

We lose analyticity

Consider functions w/ singularities

we will define a 'series'

$f(z)$ is undefined at $z = z_0, z_1$

on an annulus, instead of a disk.

$A(\alpha, r, R)$

center

inner radius

outer radius

Laurent Series: $\displaystyle\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$

$$= \underbrace{\sum_{n=1}^{\infty} c_{-n} (z-\alpha)^{-n}}_{\text{negative exponents}} + \underset{\substack{\uparrow \\ f(\alpha)?}}{c_0} + \underset{\substack{\uparrow \\ \text{positive exponents}}}{\sum_{n=1}^{\infty} c_n (z-\alpha)^n}$$

Thm: If $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ converges on $A(\alpha, r, R)$ then it converges uniformly on $\overline{A(\alpha, s, t)}$, $r<s$, $t<R$

Laurent's Thm: $0 \leq r < R$, f analytic on $A(\alpha, r, R)$. If ρ is an number $r < \rho < R$. Then $\forall z_0 \in A$

$$f(z_0) = \sum_{n=1}^{\infty} c_{-n}(z_0 - \alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z_0 - \alpha)^n$$

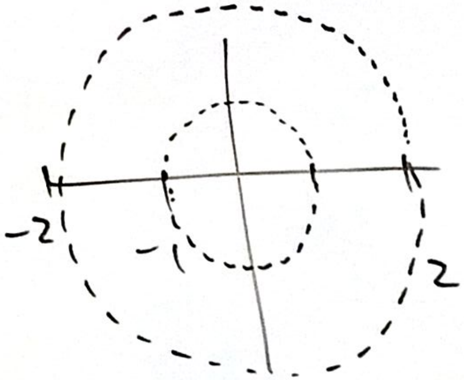
$$c_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z-\alpha)^{n+1}} dz \quad \text{for } n \geq 0$$

Claim:

$$c_{-n} = \frac{1}{2\pi i} \int \frac{f(z)}{(z-\alpha)^{-n+1}} dz$$

$$f(z) = \frac{3}{2+z-z^2}$$

has singularities at $z=-1$ and $z=2$



Laurent series at $z=0$

3 cases $|z|<1$

$1<|z|<2$

$|z|>2$

$$f(z) = \frac{1}{1+z} + \frac{1}{2-z}$$

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1+(-z)+(-z)^2+\dots \quad \text{for } |z|<1$$

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\dots\right) \quad \text{for } \left|\frac{z}{2}\right|<1 \rightarrow |z|<2$$

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k \quad |z|<1 \quad , \quad \frac{1}{1+z} = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} \quad |z|>1$$

$$\frac{1}{2-z} = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad |z|<2$$

$$\frac{1}{2+z-z^2} = \sum_{k=0}^{\infty} \underbrace{\left((-1)^k + \frac{1}{2^{k+1}}\right)}_{c_k} z^k \quad \text{for } |z|<1$$

$$\frac{1}{1+z} \quad \text{for } |z|>1$$

$$\frac{1}{1+z} = \frac{1}{z} \underbrace{\frac{1}{1+\frac{1}{z}}}_{1-\left(-\frac{1}{z}\right)} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4}$$

$$\left|\frac{1}{z}\right| < 1 \to |z|>1$$

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k \quad |z|<1$$

$$\frac{1}{1+z} = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot z^{-k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} \quad |z|>1$$

$$\frac{1}{2-z} = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad |z|<2$$

$$\frac{1}{2+z-z^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad 1<|z|<2$$

$$\frac{1}{2-z} = \frac{1}{z}\,\frac{1}{\frac{2}{z}-1} = -\frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = -\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\dots\right)$$

$$\left|\frac{2}{z}\right|<1 \to 2<|z|$$

$$\frac{1}{2-z} = \sum_{k=1}^{\infty} \frac{-2^{k-1}}{z^k}$$

$$-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}+\dots$$

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k \quad |z|<1 \qquad , \quad \frac{1}{1+z} = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot z^{-k}$$

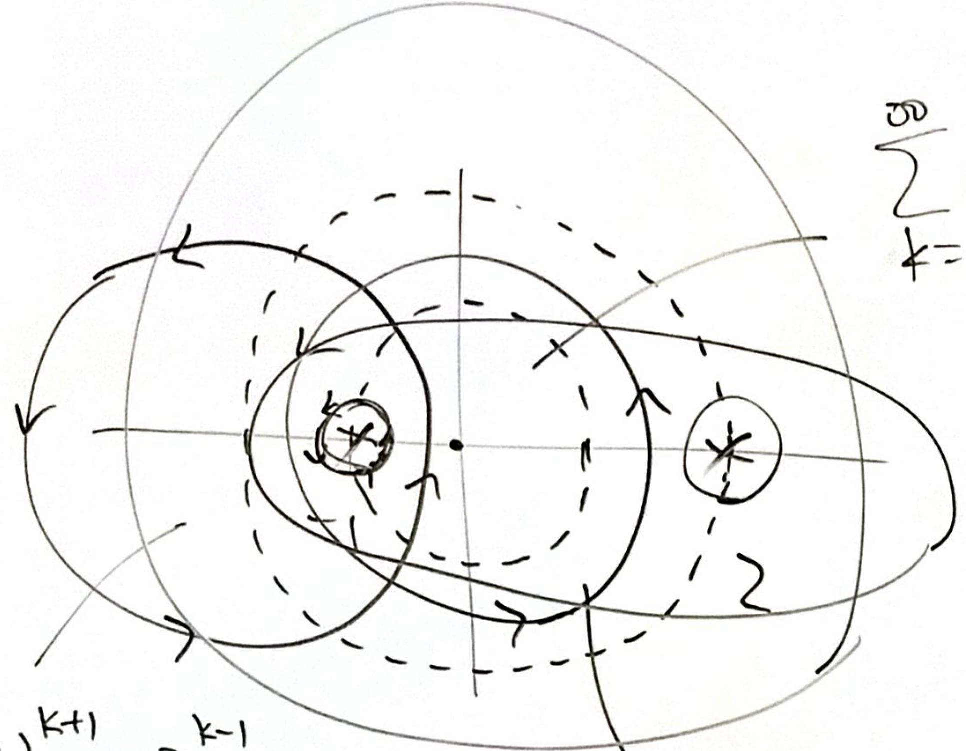
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} \quad |z|>1$$

$$\frac{1}{2-z} = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad |z|<2$$

$$\frac{1}{2+z-z^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad 1<|z|<2$$

$$\frac{1}{2+z-z^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} - 2^{k-1}}{z^k} \quad \text{for } |z|>2$$

$$\sum_{k=0}^{\infty}\left((-1)^k + \frac{1}{2^{k+1}}\right) z^k$$



$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} - 2^{k-1}}{z^k}$$

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^k} + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}$$

$$\int_{C^+_{3/2}(0)} f(z)\,dz = 2\pi i \cdot (-1)^{1+1}$$