$$\frac{1}{1-w} = 1 + w + w^2 + \cdots$$

provided $|w| < 1$

What if we want a series centered at $z = \alpha$

e.g. $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$

$$\frac{1}{1-z} = \frac{1}{1-\alpha} \cdot \frac{1}{1 - \underbrace{\frac{(z-\alpha)}{(1-\alpha)}}_{w}}$$

$$\frac{1}{1-z} = \frac{1}{1-i} \frac{1}{1 - \frac{z-i}{1-i}}$$

$$= \frac{1}{1-i}\left(1 + \frac{z-i}{1-i} + \frac{(z-i)^2}{(1-i)^2} + \cdots\right)$$

, provided

$$\left|\frac{z-i}{1-i}\right| < 1$$

or

$$|z-i| < \sqrt{2}$$

$$= \sum_{k=0}^{\infty} \frac{(z-i)^k}{(1-i)^{k+1}}$$

$$\frac{1}{1-z} = \frac{1}{1-i}\,\frac{1}{1-\frac{z-i}{1-i}} \qquad \{\boxed{\ldots, c_{-2}, c_{-1}}, \boxed{c_0, c_1, c_2 \ldots}$$

$$= \frac{1}{1-i}\,\frac{1}{z-i}\,\frac{1}{\frac{1}{z-i}-\frac{1}{1-i}}$$

$$= \frac{1}{z-i}\cdot\frac{1}{\frac{1-i}{z-i}-1}$$

$$= \frac{-1}{z-i}\,\frac{1}{1-\frac{1-i}{z-i}} \qquad \left|\frac{1-i}{z-i}\right| < 1 \;\Rightarrow\; |z-i| > \sqrt{2}$$

$$= \frac{-1}{z-i}\left(1+\frac{1-i}{z-i}+\frac{(1-i)^2}{(z-i)^2}+\cdots\right)$$

$$= -\sum_{k=0}^{\infty}\frac{(1-i)^k}{(z-i)^{k+1}} = -\sum_{k=1}^{\infty}\frac{(1-i)^{k-1}}{(z-i)^k}$$

$$\frac{1-z}{z-2} = \frac{z-1}{2-z}$$

$$= \frac{z-1}{1-(z-1)}$$

$$= z-1 \cdot \frac{1}{1-(z-1)} \qquad |z-1|<1$$

$$= z-1 \cdot \sum_{k=0}^{\infty} (z-1)^k$$

$$= \sum_{k=0}^{\infty} (z-1)^{k+1}$$

$$\sin\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{1}{z^{2k+1}}$$

Not a geometric series.

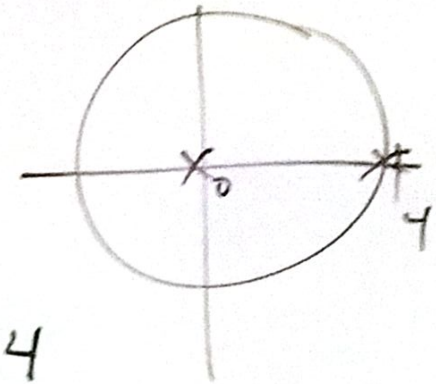
Fix $z \in \mathbb{C} - \{0\}$ $\qquad c_k = \frac{(-1)^k}{(2k+1)!}$

ratio test. $\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \cdot \frac{\frac{1}{z^{2n+3}}}{\frac{1}{z^{2n+1}}} \right|$

$$= \frac{1}{(2n+2)(2n+3)} \left| \frac{1}{z^2} \right| \to 0 \text{ as } n \to \infty$$

for $|z| > 0$

$$\frac{1}{z} \cdot \frac{1}{(4-z)^2}$$



need series for $0 < |z| < 4$

and $|z| > 4$ (w/ $\alpha = 0$)

$$\frac{1}{4-z} = \frac{1}{4}\,\frac{1}{1-\frac{z}{4}} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k \qquad |z|<4$$

$$\frac{d}{dz}\,\frac{1}{4-z} = -\frac{1}{(4-z)^2}$$

$$= -\frac{1}{z}\,\frac{1}{1-\frac{4}{z}} \qquad |z|>4$$

$$\frac{1}{(4-z)^2} = -\frac{d}{dz}\,\frac{1}{4}\sum_{k=0}^{\infty}\frac{z^k}{4^k}$$

$$f(z) = \sum_{k=1}^{\infty} c_{-k}(z-\alpha)^{-k} + c_0 + \sum_{k=1}^{\infty} c_k (z-\alpha)^k$$

if $c_{-k} = 0 \ \forall k \in \mathbb{N}$, we have a taylor series. $f(z)$ is analytic $\forall z$ on some $D_R(\alpha)$, for some $R>0$.

furthermore:

if $c_k = 0$ for $k = 0, 1, 2, \dots, n-1$, but $c_n \neq 0$

then $f(z)$ has a root of order n.

$$f(z) = (z-\alpha)^n \cdot g(z), \quad g(\alpha) \neq 0.$$

$f(z) = z^4 \sin(z)$

$f(z)$ has a root of order 5 at $z=0$,

roots of order 4 $z = n\pi$, $n = \pm 1, \pm 2, \ldots$

at $z=0$, $f(z) = z^5 \cdot \left(\underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}_{g(z)} \right)$

$g(0) \neq 0$

$$f(z) = \sum_{k=1}^{\infty} c_{-k}(z-\alpha)^{-k} + c_0 + \sum_{k=1}^{\infty} c_k (z-\alpha)^k$$

If $c_{-k} = 0$ for $k > N$,

$$f(z) = c_{-N}\frac{1}{(z-\alpha)^N} + c_{-N+1}\frac{1}{(z-\alpha)^{n-1}} + \cdots + c_{-1}\frac{1}{z-\alpha} +$$

we say $f(z)$ has a <u>pole</u> of order N.

IF $N = \infty$, we say $z = \alpha$ is an essential singularity.

Essential Singularity:

$$\sin\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{z^{2k+1}}$$

has an essential singularity at $z=0$

$$f(z) = \frac{\sin(z)}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

has a simple pole at $z=0$.