8.1 The Residue Theorem

Let f have a isolated removable singularity

at the pole $z = z_0$.

thus $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ for some $n \in \mathbb{N}$

w/ $g(z_0) \neq 0$ and analytic at $z = z_0$.

Then the <u>residue</u> of $f(z)$ at $z = z_0$ is

the -1 coefficient in the Laurent Series

$$f(z) = \sum_{k=-n}^{\infty} a_n (z - z_0)^n$$

$$\text{Res}[f, z_0] = a_{-1}$$

Ex: $g(z) = \dfrac{3}{2z + z^2 - z^3}$

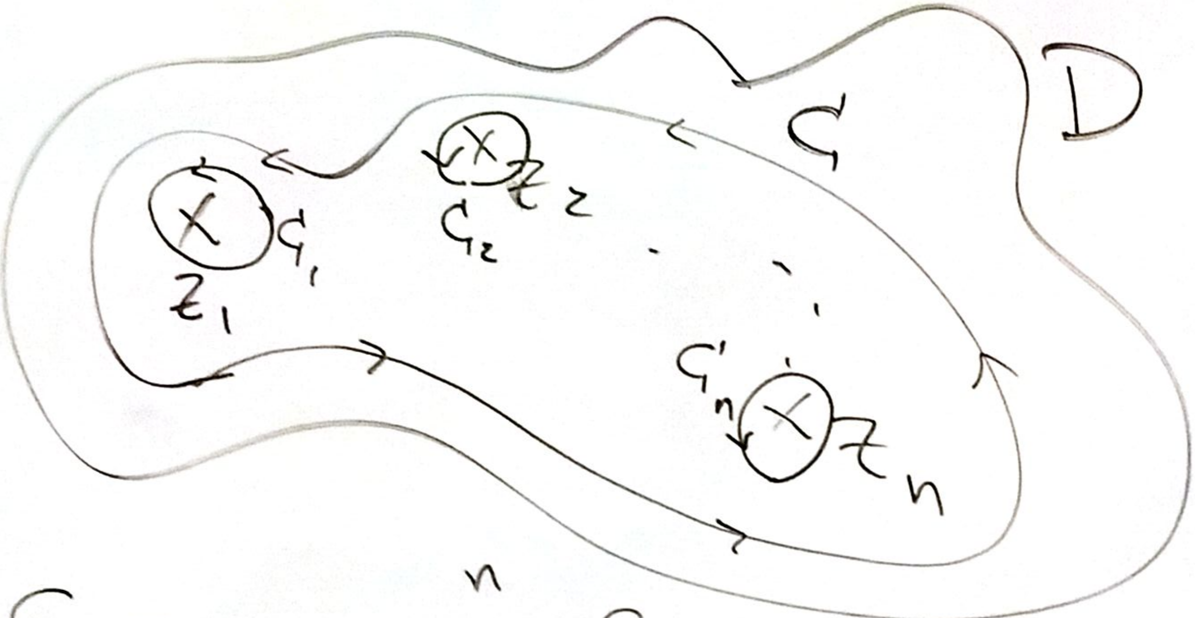
Find $\text{Res}[g, 0] = \dfrac{3}{2}$

$$g(z) = \frac{3}{2}\frac{1}{z} - \frac{3}{4} + \frac{9}{8}z - \frac{15}{16}z^2 \ldots$$

Cauchy's Residue Theorem

Let D be simply connected, C be a positively oriented contour s.t. $C \subset D$, C is simple closed. If $f(z)$ is analytic inside C and on C except at $z_1, z_2, \ldots, z_n$ $\operatorname{int}(C)$, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f, z_k]$$



$$\int_{C_1} f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{n} \int_{C_k} \sum_{r=-l_k}^{\infty} a_r (z-z_k)^r$$

$$2\pi i \sum_{k=1}^{n} a_{k,-1}$$

$$= 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f, z_k]$$

How to find residues

See page 275

Claim: for a simple pole:

$$\text{Res}[f, z_0] = \lim_{z \to z_0} \left((z - z_0) \cdot f(z) \right)$$

By def of simple pole:

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \ldots$$

$$(z - z_0) \cdot f(z) = a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2$$

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1}$$

Pole of order 2:

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots$$

we cannot take $\lim_{z \to z_0}$ yet.

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = 0 + a_{-1} + 2a_0(z-z_0) + 3\ldots$$

$$\lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1}$$

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pole of order n:

$$\text{Res}[f, z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \right]$$

Ex: $\displaystyle\int_{C_3^+(0)} \frac{1}{z^4+z^3-2z^2}\, dz$

$$z^4+z^3-2z^2 = z^2(z+2)(z-1)$$

3 poles, $z = 0, -2, 1$ of orders $2, 1, 1$ respectively

$$z_1 = 0,\ z_2 = -2,\ z_3 = 1 \in \mathrm{Int}(C_3^+(0))$$

$$\int_{C_3^+(0)} f(z)\,dz = \int_{C_{\varepsilon_1}^+(0)} f(z)\,dz + \int_{C_{\varepsilon_2}^+(1)} f(z)\,dz + \int_{C_{\varepsilon_3}^+(-2)} f(z)\,dz$$

$$0 < \varepsilon_{1,2,3} < 1$$

$$= 2\pi i \left(\mathrm{Res}[f,0] + \mathrm{Res}[f,1] + \mathrm{Res}[f,-2] \right)$$

$$\mathrm{Res}[f,1] = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{1}{z^2(z+2)\cancel{(z-1)}} = \frac{1}{3}$$

$$\mathrm{Res}[f,-2] = \lim_{z\to -2} \cancel{(z+2)} \cdot \frac{1}{z^2(z-1)\cancel{(z+2)}} = -\frac{1}{12}$$

$$z_1 = 0,\ z_2 = -2,\ z_3 = 1 \in \mathrm{Int}(C_3^+(0))$$

$$\int_{C_3^+(0)} f(z)\,dz = \int_{C_{\varepsilon_1}^+(0)} f(z)\,dz + \int_{C_{\varepsilon_2}^+(1)} f(z)\,dz + \int_{C_{\varepsilon_3}^+(-2)} f(z)\,dz$$

$$0 < \varepsilon_{1,2,3} < 1$$

$$= 2\pi i \left(\mathrm{Res}[f,0] + \underbrace{\mathrm{Res}[f,1]}_{\frac{1}{3}} + \underbrace{\mathrm{Res}[f,-2]}_{-\frac{1}{12}} \right)$$

$$\mathrm{Res}[f,0] = \frac{1}{1!} \lim_{z \to 0} \left[\frac{d}{dz} \cancel{z^2} \cdot \frac{1}{\cancel{z^2}(z+2)(z-1)} \right]$$

$$= \lim_{z \to 0} -\frac{2z+1}{(z^2+z-2)^2} = -\frac{1}{4}$$

$$z_1 = 0,\ z_2 = -2,\ z_3 = 1 \in \mathrm{Int}(C_3^+(0))$$

$$\int_{C_3^+(0)} f(z)\,dz = \int_{C_{\varepsilon_1}^+(0)} f(z)\,dz + \int_{C_{\varepsilon_2}^+(1)} f(z)\,dz + \int_{C_{\varepsilon_3}^+(-2)} f(z)\,dz$$

$0 < \varepsilon_{1,2,3} < 1$

$$= 2\pi i \left(\mathrm{Res}[f, 0] + \mathrm{Res}[f, 1] + \mathrm{Res}[f, -2] \right)$$

$$= 2\pi i \left(-\frac{1}{4} + \frac{1}{3} - \frac{1}{12} \right)$$

$$= 0$$

Look at Ex 8.7