8.1

3(c) $\displaystyle\int_{C_2^+(0)} \underbrace{\frac{e^z}{z^3+z}}_{f} \, dz$ poles of order 1 at $z=0, \pm i$

$\displaystyle Res[f,0] = \lim_{z\to 0} \cancel{z} \cdot \frac{e^z}{\cancel{z}(z+i)(z-i)} = 1$

$\displaystyle Res[f,i] = \lim_{z\to i} \cancel{(z-i)} \cdot \frac{e^z}{z\cdot(z+i)\cancel{(z-i)}} = -\frac{e^i}{2}$

$\displaystyle Res[f,-i] = \lim_{z\to -i} \cancel{(z+i)} \frac{e^z}{z\cancel{(z+i)}(z-i)} = -\frac{e^{-i}}{2}$

$\displaystyle = 2\pi i\left(1 - \frac{e^{i\cdot 1}+e^{-i\cdot 1}}{2}\right) = 2\pi i(1-\cos(1))$

#4 Let $f, g$ be analytic at $z_0$.
$f(z_0) \neq 0$, $g(z)$ has a simple root
at $z = z_0$.

Show $\text{Res}\left[\frac{f}{g}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$

$= \lim_{z \to z_0} (z - z_0) \cdot \frac{f(z)}{g(z)}$

$g(z) = (z - z_0)h(z)$ w/ $h(z_0) \neq 0$
$g'(z) = h(z) + (z - z_0)h'(z)$

#4 $\lim_{z \to z_0} (z-z_0) \frac{f(z)}{g(z)}$

$= \lim_{z \to z_0} \frac{\cancel{(z-z_0)} f(z)}{\cancel{(z-z_0)} h(z)}$

$= \frac{f(z_0)}{h(z_0)} = \frac{f(z_0)}{g'(z_0)}$

$g(z) = (z-z_0)h(z)$ w/ $h(z_0) \neq 0$

$g'(z) = h(z) + (z-z_0)h'(z)$

$$\int_0^{2\pi} \frac{1}{1+3\cos^2(\theta)}\, d\theta$$

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin(z) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

and $C_1 = C_1^+(0)$

$$\int_{C_1^+(0)} P(z)\, dz$$

$$dz = ie^{i\theta}\, d\theta$$

$$z = e^{i\theta} \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\int_0^{2\pi} \frac{1}{1+3\cos^2(\theta)}\, d\theta$$

$$z = e^{i\theta} \longrightarrow dz = i\overbrace{e^{i\theta}}^{z}\, d\theta, \quad d\theta = \frac{dz}{iz}$$

$$\oint_{C_1^+(0)} \frac{1}{1+3\left(\frac{z+\frac{1}{z}}{2}\right)^2} \frac{dz}{iz} \xleftarrow{\text{let}} z = e^{i\theta}$$

$\theta : 0 \longrightarrow 2\pi$

$$\int_{C_1^+(0)} \frac{-i4z}{3z^4+10z^2+3}\,dz \qquad f(z)$$

$z = \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}$ lie inside unit circle

$$2\pi i\left(\text{Res}\left[f, \frac{i}{\sqrt{3}}\right] + \text{Res}\left[f, -\frac{i}{\sqrt{3}}\right]\right)$$

$$= 2\pi i\left[-\frac{i}{4} - \frac{i}{4}\right] = \pi$$

$$= \int_0^{2\pi} \frac{1}{1+3\cos^2(\theta)}\,d\theta$$

$$\int_0^{2\pi} F(\cos(\theta), \sin(\theta))\, d\theta$$

$$\int_{C_1^+(0)} F\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) \frac{dz}{iz}$$

$$z = e^{i\theta} \qquad dz = i e^{i\theta}\, d\theta$$

$$= 2\pi i \cdot \sum \text{residues inside } C_1(0)$$

$$\int_0^{2\pi}$$

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$z = re^{i\theta} \qquad dz = ire^{i\theta}$$

$$r\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) =$$

$$r\cos(\theta) \neq \frac{z+\frac{1}{z}}{2}$$

$$\int_0^{2\pi} \frac{\cos(2\theta)}{5+3\cos(\theta)}\,d\theta$$

$$\cos(\theta) = \frac{z+\frac{1}{z}}{2} \quad \text{if } z = e^{i\theta}$$

what is $\cos(2\theta)$?

$$z^2 = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$$

$$z^{-2} = e^{-i2\theta} = \cos(2\theta) - i\sin(2\theta)$$

$$\cos(2\theta) = \frac{z^2 + \frac{1}{z^2}}{2}$$

$$\int_0^{2\pi} \frac{\cos(2\theta)}{5+3\cos(\theta)}\,d\theta$$

Remember:

$z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

or $d\theta = \frac{dz}{iz}$

$$\int_{C_1^+(0)} \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{5+\frac{3}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \int_{C_1^+(0)} \frac{-i(z^4+1)}{z^2(3+z)(1+3z)}\,dz$$

order 2

order 1

$$= 2\pi i\cdot\left(\text{Res}[f,0] + \text{Res}[f,-1/3]\right)$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)^2 h(z)}$$

w/ $h(z_0) \neq 0$

$$\mathrm{Res}\left[\frac{f}{g}, z_0\right] = \lim_{z \to z_0} \frac{d}{dz} \frac{(z-z_0)^2 \cdot f(z)}{(z-z_0)^2 h(z)}$$

$$= \lim_{z \to z_0} \frac{d}{dz} \frac{f(z)}{h(z)}$$

$=$