Math 2283 - Honors Logic

Homework - Chapter 7

Name: ______________________________

Propositional System Ξ:

Symbols

The system uses lower case letters such as p, q, and r, to represent propositional symbols.

The symbol $\vee$ is the disjunctive connective: $p \vee q$ means 'p or q'.

$\overline{}$ is the symbol for negation: $\overline{p}$ is the negation of sentence p, while $\overline{p \vee q}$ is the negation of the sentence $p \vee q$.

Well Formed Formulas (WFFs)

(1) Any propositional symbol is a wff.
(2) If A is a wff, then $\overline{A}$ is a wff.
(3) If A and B are wffs, then $A \vee B$ is a wff.
(4) Nothing else is a wff.

Examples:

(1) p is a wff.
(2) $\overline{p} \vee q$ is a wff.
(3) $\overline{\overline{p \vee q} \vee r}$ is a wff.
(4) $\overline{\overline{p \vee q}\vee} r$ is not a wff.

Definition: A wff A is a *primitive disjunction* iff A is of the form $B_1 \vee B_2 \vee \ldots \vee B_n$ where each B_i is either a propositional symbol or a propositional symbol with a bar over it (i.e. each B_i is of the form p or $\overline{p}$).

Examples:

(1) $p \vee \overline{q} \vee r$ is a primitive disjunction.
(2) $\overline{p \vee \overline{q}}$ is not a primitive disjunction.
(3) $\overline{\overline{p}} \vee q$ is not a primitive disjunction.
(4) $\overline{p} \vee \overline{q} \vee q \vee p$ is a primitive disjunction.

Definition: Every wff A is a *disjunctive part* of itself, and if A is a *disjunctive part* of $B \vee C$, then B is a *disjunctive part* of A and so is C.

Examples:

(1) p is a disjunctive part of p.
(2) $\overline{p}$ is a disjunctive part of $q \vee \overline{p} \vee p$.
(3) $\overline{p} \vee p$ is a disjunctive part of $q \vee \overline{p} \vee p$.
(4) $q \vee p$ is a not disjunctive part of $q \vee \overline{p} \vee p$.

Notation: $D(A)$ is a wff of which A is a disjunctive part, and $D(B)$ is the result of replacing *one occurrence* of the disjunctive part A in $D(A)$ by B.

Examples:

(1) If $D(p)$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $q \vee \overline{r} \vee \overline{p} \vee q$.
(2) If $D(q)$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $p \vee \overline{p} \vee q \vee \overline{r}$.
(3) If $D(\overline{p})$ is $p \vee \overline{p} \vee q$, then $D(q \vee \overline{r})$ is $p \vee q \vee \overline{r} \vee q$.
(4) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is not $\overline{r} \vee \overline{p} \vee \overline{r} \vee r$.
(5) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is $p \vee \overline{p} \vee \overline{r} \vee r$.
(6) If $D(p)$ is $p \vee \overline{p} \vee p \vee r$, then $D(\overline{r})$ is $\overline{r} \vee \overline{p} \vee p \vee r$.

Axioms of System Ξ:

There are an infinite number of axioms for System Ξ. A wff A is an axiom iff A is a primitive disjunction and for some propositional symbol B, both B and $\overline{B}$ are disjunctive parts of A.

Examples:

(1) $q \vee \overline{r} \vee \overline{q} \vee p$ is an axiom.
(2) $q \vee \overline{r} \vee \overline{q \vee \overline{r}}$ is not an axiom.
(3) $\overline{\overline{q}} \vee \overline{q}$ is not an axiom.
(4) $q \vee \overline{r} \vee \overline{q} \vee \overline{q \vee \overline{r}}$ is not an axiom.
(5) $\overline{q} \vee r \vee p \vee \overline{q} \vee \overline{r}$ is an axiom.

Rules of Inference for System Ξ

Rule I: $D(\overline{\overline{A}})$ is an immediate consequence of $D(A)$.
Rule II: $D(\overline{A \vee B})$ is an immediate consequence of $D(\overline{A})$ and $D(\overline{B})$

Examples:

(1) $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
(2) $p \vee \overline{\overline{\overline{r}}} \vee \overline{p} \vee q$ is an immediate consequence of $p \vee \overline{r} \vee \overline{p} \vee q$ by Rule I.
(3) $\overline{\overline{p} \vee q} \vee \overline{r} \vee \overline{p} \vee q$ is an immediate consequence of $\overline{\overline{p}} \vee \overline{r} \vee \overline{p} \vee q$ and $\overline{q} \vee \overline{r} \vee \overline{p} \vee q$ by Rule II.
(4) $q \vee \overline{\overline{p} \vee q} \vee \overline{p} \vee r$ is an immediate consequence of $q \vee \overline{\overline{p}} \vee \overline{p} \vee r$ and $q \vee \overline{q} \vee \overline{p} \vee r$ by Rule II.

There exists a method of proof in System Ξ to derive any tautological sentence which can be aided by the construction of a tree. An example of this method is given in the two trees below for the sentence $\overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p$. Note that the trees are simply reflections of each other, one growing downwards, the second growing upwards.

$$
\begin{array}{ccc}
 & \overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p & \\
\swarrow & & \searrow \\
\overline{\overline{\overline{p}}} \vee \overline{q} \vee p & & \overline{\overline{q}} \vee \overline{q} \vee p \\
| & & | \\
\overline{p} \vee \overline{q} \vee p & & q \vee \overline{q} \vee p
\end{array}
$$

The first tree (which grows downwards) is used to dissect the original sentence using Rules I and II and the axioms given. Each branch of the tree terminates at an axiom. Notice, in the lowest branch of the above tree, the two sentences $\overline{p} \vee \overline{q} \vee p$ and $q \vee \overline{q} \vee p$ are axioms, as the first contains both p and $\overline{p}$, while the second contains both q and $\overline{q}$.

To build the tree, work from left to right and pick the leftmost portion of the wff which is not in primitive disjunctive form. For instance, $\overline{\overline{\overline{p}} \vee \overline{q}}$ is the only portion of the original sentence not in primitive disjunctive form. Only an application of Rule II would yield $\overline{\overline{\overline{p}} \vee \overline{q}}$ as part of a wff, so we simply determine the two wffs so that when Rule II is applied to them, we get $\overline{\overline{\overline{p}} \vee \overline{q}}$. The two wffs required are $\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$ and $\overline{\overline{q}} \vee \overline{q} \vee p$.

At each stage, we check to see if the wffs used are primitive disjunctions. In this example, they are not, but it can be easily seen that Rule I can be used on $\overline{p} \vee \overline{q} \vee p$ to arrive at $\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$, and can also be used on $q \vee \overline{q} \vee p$ to arrive at $\overline{\overline{q}} \vee \overline{q} \vee p$. Now every branch in the tree above ends in a primitive disjunction. Furthermore, they are axioms and thus we can build a proof to generate the sentence in question from axioms using only Rules I and II. To make it easier to see this, we invert the tree as below and build the proof.

$$
\begin{tikzcd}[column sep=small]
\overline{p} \vee \overline{q} \vee p \arrow[d, dash] & & q \vee \overline{q} \vee p \arrow[d, dash] \\
\overline{\overline{\overline{p}}} \vee \overline{q} \vee p \arrow[dr, dash] & & \overline{\overline{q}} \vee \overline{q} \vee p \arrow[dl, dash] \\
& \overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p &
\end{tikzcd}
$$

The following is the proof generated by the tree given above. Note that only axioms, and Rules I and II are used. This is the approach to be taken for any wff in this system which you believe is a tautology.

(1)	$q \vee \overline{q} \vee p$	Axiom
(2)	$\overline{p} \vee \overline{q} \vee p$	Axiom
(3)	$\overline{\overline{q}} \vee \overline{q} \vee p$	Rule I applied to (1)
(4)	$\overline{\overline{\overline{p}}} \vee \overline{q} \vee p$	Rule I applied to (2)
(5)	$\overline{\overline{\overline{p}} \vee \overline{q}} \vee \overline{q} \vee p$	Rule II applied to (4) and (3)

$\square$

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1. Explain why the conditions stated to be an axiom of System Ξ imply that every axiom is a tautology.

2. Construct arguments that Rules I and II of System Ξ indeed preserve the notion of a tautology. I.e. if Rule I is applied to a tautological sentence A, then the result is also a tautological sentence. Similarly, if Rule II is applied to two tautological sentences A and B, the result is also a tautology. You do not need a rigorous proof, but your argument should be convincing.

3. Is System Ξ simply and/or absolutely consistent? Explain your answer thoroughly.

4. Is System Ξ semantically complete with respect to the axioms and two rules of inference previously specified? Explain your answer thoroughly.

5. Express the logical sentence $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$ as a valid sentence of System Ξ. I.e. write it in terms of negations and disjunctions in such a way that it is a wff of System Ξ.

6. Create a tree for the wff you constructed in the previous problem.

7. Using the tree from the previous problem, construct a proof using only axioms, and Rules I and II, that shows your wff is a tautology.

8. Starting with the sentence $(p \leftrightarrow q) \rightarrow \{[(r \wedge p) \rightarrow s] \rightarrow [(r \wedge q) \rightarrow s]\}$, repeat the work done in problems 5, 6, and 7. I.e. convert the given logical law to a wff of System Ξ and prove it is a tautology in the same manner as you did for $[(p \vee q) \wedge (p \rightarrow r)] \rightarrow (r \vee q)$.